A systematic stability analysis of the renormalisation group flow for the normal-superconductor-normal junction of Luttinger liquid wires

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We study the renormalization group flows of the two terminal conductance of a superconducting junction of two Luttinger liquid wires. We compute the power laws associated with the renormalization group flow around the various fixed points of this system using the generators of the $SU(4)$ group to generate the appropriate parameterization of an $\mathbb{S}$-matrix representing small deviations from a given fixed point $\mathbb{S}$-matrix (obtained earlier in Phys. Rev. **B 77**, 155418 (2008)), and we then perform a comprehensive stability analysis. In particular, for the non-trivial fixed point which has intermediate values of transmission, reflection, Andreev reflection and crossed Andreev reflection, we show that there are eleven independent directions in which the system can be perturbed, which are relevant or irrelevant, and five directions which are marginal. We obtain power laws associated with these relevant and irrelevant perturbations. Unlike the case of the two-wire charge-conserving junction, here we show that there are power laws which are non-linear functions of $V(0)$ and $V(2k_F)$ (where $V(k)$ represents the Fourier transform of the inter-electron interaction potential at momentum k). We also obtain the power law dependence of linear response conductance on voltage bias or temperature around this fixed point.

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Electron-electron (e-e) interactions in low-dimensional systems (one-dimensional (1–D) quantum wires (QW) and dots) can lead to non-trivial low energy transport properties due to the Luttinger liquid (LL) ground state of the system. In this context, a geometry which has gained considerable attention in the recent past is the multiple LL wire junction. In general, junctions of multiple QW can be viewed as quantum impurities in a LL from which electrons get scattered at the junction. For the simplest case of two-wires, the junction can be modeled as a back-scatterer while for the general case of multiple QW, the junction represents a more non-trivial quantum impurity which may not be as straightforward to model microscopically.

For the two-wire system, it is well-known[1,2] that in the presence of a scatterer, there are only two low energy fixed points - *(i) the disconnected fixed point* with no transmission (i.e. the transmission amplitude for incident electron or hole, $t = 0$) which is stable and *(ii) the transmitting fixed point* with no reflection ($t = 1$) which is unstable. More recently, the low energy dynamics of multiple LL wires connected to a junction have also been studied in detail[3,4,5,6,7,8,9,10] and several interesting fixed points have been found, including continuous one-parameter families of fixed points[11]. These studies have also been generalized theoretically[12,13,14,15,16,17,18,19,20] to describe a junction of 1–D wires with superconductors and have also been generalized to include spin[21]. Our recent work[22,23,24] has generalized these studies to the case of superconducting junction of multiple QW. In such a system, due to the proximity of the superconductor, both electrons as well as holes take part in the transport which leads to very interesting transport properties at small bias resulting from the interplay of LL correlations and the proximity induced pair potential.

In this article, we study the case where two LL QW are coupled simultaneously to a bulk superconductor. The physical separation between the junctions of the two-wires with the superconductor is of the order of the size of the Cooper-pair. This leads to the realization of a normal-superconductor-normal (NSN) junction which allows for direct tunneling of electrons from one wire to the other and also allows a finite amplitude for the crossed Andreev reflection (CAR) process[25,26,27,28,29,30,31,32,33,34,35] in addition to the normal reflection and Andreev reflection (AR) processes.

In an earlier study of the NSN junction[23], we showed that the NSN junction has more than two fixed points unlike the normal two-wire junction (as mentioned above) or the junction of LL with a bulk superconductor (NS junction) which has only two fixed points - *(i) the Andreev fixed point* where the amplitude for Andreev reflection (AR), $r_A = 1$ and normal reflection amplitude, $r = 0$ and which is unstable and *(ii) the disconnected fixed point* where $r_A = 0$ and $r = 1$, and which is stable[12,13]. We showed that there exists a fixed point with intermediate values of transmission and reflection. Thus, the NSN junction is the minimum configuration which possesses non-trivial fixed points with intermediate transmission and reflection amplitudes. In what follows, we will focus mainly on the NSN junction.

In the previous studies, a comprehensive analysis of the various possible perturbations allowed by symmetry around all the fixed points of the NSN junction was lacking. In this article, we carry out a systematic stability analysis for each of the fixed points obtained earlier in Ref. 23 for the NSN junction and we predict the power laws associated with all possible independent perturbations that can be switched on around these fixed points. Our analysis provides us with renormalized values of the various transmission and reflection amplitudes around these fixed point values which can then be used to obtain the Landauer-Büttiker conductances.

We start with a brief review of the RG method followed in Refs. 2 and 4 where an $\mathbb{S}$-matrix formulation was used to compute the linear conductance and inter-electron interactions inside the QW were taken into account by allowing the $\mathbb{S}$-matrix to flow as a function of the relevant energy scale (like temperature, bias voltage or system size) using an RG procedure. This method works well when e-e interaction strength inside the QW is weak so that it can be treated perturbatively. This is usually referred to as the weak interaction renormalization group (WIRG) procedure[36].

To benchmark our calculation with known results, we first calculate the complete set of all possible power laws associated with the independent perturbations that can be switched on around the time-reversal symmetry broken chiral fixed points (CFP) and the time reversal symmetric Griffiths fixed point (GFP) of a normal junction of three LL wires[4]. Then we apply the same procedure to the case of the NSN junction. The strength of our formulation to obtain the power law scaling of the perturbations turned on around the various fixed points lies in the fact that the same method is applicable to both normal as well as superconducting junctions of any number of QW.

Now, for the case of three LL wires meeting at a normal junction, let us assume that the wires are parameterized by spatial coordinates x_i which go from zero to infinity with the junction being situated at $x_i = 0$ (i being the wire index). The junction can be parameterized by a 3×3 $\mathbb{S}$-matrix with diagonal elements r_{ii} and off-diagonal elements t_{ij}. In the presence of e-e interactions, the RG equations can be derived[2,4] by first expanding the electron wave-function in each of the wires in terms of reflected and transmitted electron waves (scattering wave basis). Then the amplitude of scattering of electrons from the Friedel oscillations in the wires can be deduced by using a Hartree-Fock decomposition of the e-e interaction term. Finally the RG equation is obtained by applying the poor-man's scaling approach[36]. The RG equations for the entire $\mathbb{S}$-matrix can be written in a concise and compact from given by[4]

$$\frac{d\mathbb{S}}{dl} = \mathbb{F} - \mathbb{S}\mathbb{F}^{\dagger}\mathbb{S} \ , \tag{1}$$

where, $l = \ln(L/d)$ is the dimension-less RG scale (L corresponds to the physical length scale or energy scale at which we are probing the system and d is the short distance or high energy cut-off). Here $\mathbb{F}$ is a diagonal matrix with $\mathbb{F}_{ii} = -\alpha r_{ii}/2$ and α is the repulsive e-e interaction parameter which is related to the LL parameter K as $K = ((1-\alpha)/(1+\alpha))^{1/2}$.

Analogously, the NSN junction can be described in terms of an $\mathbb{S}$-matrix with elements describing transmission of both electrons and holes and their mixing at the junction. The corresponding RG equation for the $\mathbb{S}$-matrix was obtained by the present authors[22,23] which was an extension of Eq. 1 to the superconducting case. Here too for the NSN case, we will assume that the two wires are parameterized by spatial coordinates x_1 and x_2 where x_1, x_2 vary from zero to infinity and that the junction is at $x_1 = 0 = x_2$. The presence of the superconductor is encoded in the parametrization of the $\mathbb{S}$-matrix representing the junction. Of course, this way of accounting for the presence of the superconductor assumes that the superconductor imposes static boundary conditions on the two wires forming the NSN junction. This is a valid approximation as long as one is focusing on sub-gap transport properties of the junction. We also assume that the superconductor at the junction is a singlet superconductor; hence the spin of the incident electron or hole is conserved as it scatters off the junction. This results in a block diagonal form of the $\mathbb{S}$-matrix with each spin block being a 4×4 matrix representing scattering of electrons and holes within the given spin sector.

The $\mathbb{S}$-matrix at the superconducting junction for the spin-up, spin-down, electron-hole and left-right symmetric (symmetry in wire index) case (suppressing the wire index) can be parameterized by r, the normal reflection amplitude, r_A, the AR amplitude, t_A, the CAR amplitude[25,26] and t, the transmission amplitude for both electrons and holes. The fermion fields can then be expanded around left and right Fermi points on each wire as $\psi_{is}(x) = \Psi_{I\,is}(x)\, e^{i\,k_F\,x} + \Psi_{O\,is}(x)\, e^{-i\,k_F\,x}$; where i is the wire index, s is the spin index which can be $\uparrow, \downarrow$ and $I(O)$ stands for incoming (outgoing) fields. Note that $\Psi_{I(O)}(x)$ are slowly varying fields on the scale of k_F^{-1}. Electrons with momenta k in vicinity of k_F, on each wire at position x is given by

$$\begin{aligned}\Psi_{is}(x) \;=\; & \int_0^\infty dk \Big[b_{ks} e^{i\,(k+k_F)\,x} + d_{ks}^\dagger e^{i\,(-k+k_F)\,x} \\ & + \; r b_{ks} e^{-i\,(k+k_F)\,x} + r^\star d_{ks}^\dagger e^{-i\,(-k+k_F)\,x} \\ & + \; r d_{ks} e^{-i\,(-k+k_F)\,x} + r^\star b_{ks}^\dagger e^{-i\,(k+k_F)\,x} \Big]\end{aligned} \tag{2}$$

where b_{ks} is the particle destruction operator and d_{ks} is the hole destruction operator and we have allowed for non-conservation of charge due to the proximity effect induced by the superconductor. We then allow for short-range density-density interactions between the fermions,

$$\mathcal{H}_{\mathsf{int}} = \frac{1}{2} \int dx\, dy\, (\sum_s \rho_{is}) V(x-y) (\sum_{s'} \rho_{is'}) \ , \tag{3}$$

Following the procedure outlined in Ref. 23, we find that the RG equation for the $\mathbb{S}$-matrix continues to be of the form given in Eq. 1, but now $\mathbb{F}$ is a non-diagonal matrix,

$$\mathbb{F} = \begin{bmatrix} \alpha r/2 & 0 & -\alpha' r_A/2 & 0 \\ 0 & \alpha r/2 & 0 & -\alpha' r_A/2 \\ -\alpha' r_A/2 & 0 & \alpha r/2 & 0 \\ 0 & -\alpha' r_A/2 & 0 & \alpha r/2 \end{bmatrix} \tag{4}$$

where α and α' are the x-independent part of the mean field amplitudes for Friedel oscillations and the proximity induced pair potential inside the QW respectively. Generalization to particle-hole non-symmetric situations will make this matrix asymmetric. It is worth pointing out that even though the expression for electron field in Eq. 2 assumes particle-hole symmetry which leads to considerable simplification in the derivation for the RG equation (Eq. 1), our formalism is more general. The RG equation (with appropriate modification of the $\mathbb{F}$ matrix for the asymmetric case) will also hold for $\mathbb{S}$-matrices representing situations where the wire index symmetry as well as the particle-hole symmetry is broken.

We will mainly focus on three different fixed points - the CFP and GFP of a normal junction of three LL wires[4,37] and the symmetric fixed point (SFP)[23] of the NSN junction. First we discuss the stability around the CFP and GFP of a normal junction of three LL wires (Y-junction) to benchmark our calculation with known results[4]. As a first step towards performing a systematic stability analysis, we need to obtain an $\mathbb{S}$-matrix which results from a very small unitary deviation from the fixed point $\mathbb{S}$-matrix. Given the number of independent parameters of the $\mathbb{S}$-matrix dictated by symmetry and unitarity constraints, the most general deviation from the fixed point $\mathbb{S}$-matrix can be obtained by multiplying the fixed point $\mathbb{S}$-matrix by another unitary matrix which is such that it allows for a straightforward expansion in terms of small parameters around the identity matrix. This is realized as follows -

$$\mathbb{S} = \mathbb{S}_0 \exp\left\{ i \sum_{j=1}^{9} \epsilon_j \lambda_j \right\} \,, \tag{5}$$

where $\mathbb{S}_0$ represents the fixed point $\mathbb{S}$-matrix and λ_j's (along with the identity $\lambda_0 = I$) are the eight generators of the $SU(3)$ group which are traceless hermitian matrices. This can be straightforwardly generalized to the case of N wires by using $SU(N)$ matrices. Perturbations around these fixed points are characterized in terms of the ϵ_j's. Of course, the resulting $\mathbb{S}$-matrix obtained in this way corresponds to a small unitary deviation when ϵ_j's are small. To first order in ϵ_j's, Eq. 5 reduces to

$$\mathbb{S} = \mathbb{S}_0 \left(\mathbb{I} + i \sum_j \epsilon_j \lambda_j \right) \,, \tag{6}$$

where $\mathbb{S}_0$ for CFP and GFP fixed points are[4]

$$\mathbb{S}_0^{\mathsf{CFP}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \; ; \; \mathbb{S}_0^{\mathsf{GFP}} = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \,, \tag{7}$$

respectively. Using Eq. 6, the RG equation (Eq. 1) with $\mathbb{S}$ expanded to the linear order in ϵ_j becomes

$$i \sum_{j=1}^{9} \lambda_j \frac{d\epsilon_j}{dl} \;=\; \mathbb{S}_0^\dagger \Big[\mathbb{I} - i \sum_j \epsilon_j \lambda_j\Big] \Big\{ \mathbb{F} - \mathbb{S}_0 \Big[\mathbb{I} + i \sum_j \epsilon_j \lambda_j\Big] \mathbb{F}^\dagger \Big[\mathbb{I} + i \sum_j \epsilon_j \lambda_j\Big] \Big\} \,, \tag{8}$$

where $\mathbb{F}$ is the diagonal part of the following quantity

$$\mathbb{F} = \frac{\alpha}{2} \, \mathbb{S}_0 \Big[\mathbb{I} + i \sum_j \epsilon_j \lambda_j\Big]_{\mathrm{diagonal}} \,. \tag{9}$$

Restricting the RHS of Eq. 8 to linear order in ϵ_j's, one then obtains nine coupled linear differential equations. Next by applying a unitary rotation, we can decouple these coupled equations (Eq. 8) and re-cast them in terms of new variables ϵ'_j (which are linear combinations of the original ϵ_j). The equations are now given by

$$\frac{d\epsilon'_j}{dl} = \mu_j \, \epsilon'_j \tag{10}$$

where μ_j is a real number corresponding to the 'power law' associated with perturbations turned on along each of the new nine eigen-directions ϵ'_j. $\mu_j < 0$ indicates that the given direction is stable and $\mu_j > 0$ indicates that it is unstable. Here the non-diagonal ϵ_j are related to the diagonal ϵ'_j by $\epsilon_j = \sum_i \mathbb{U}_{ji} \, \epsilon'_i$ where $\mathbb{U}$ is the diagonalizing rotation matrix.

Hence we obtain all the power laws associated with the independent perturbations that can be switched on around a given fixed point $\mathbb{S}$-matrix. Now it is straightforward to show that the power laws associated with the CFP and GFP are given by [$\alpha/2$, $\alpha/2$, 0, $\alpha/2$, $\alpha/2$, $\alpha/2$, $\alpha/2$, 0, 0] and [0, 0, 0, 0, 0, -$\alpha/3$, $2\alpha/3$, $2\alpha/3$, α] respectively which is consistent with results obtained in Ref. 4[41]. The value zero corresponds to marginal directions while the values with positive or negative signs correspond to stable or unstable directions respectively. We do not write the explicit form of the $\mathbb{U}$ matrix for the CFP and GFP as they are needed only for obtaining the explicit form of the power law correction to the fixed point conductance which we do not calculate for these cases.

Finally, let us discuss the stability around the different RG fixed points of the NSN junction. First we focus on the SFP of the NSN junction. In the presence of e-e interaction inside the QW , the incident electron (hole) not only scatters from the Friedel oscillations as an electron (hole) but also scatters from the proximity induced pair potential inside the QW as a hole (electron). Now the amplitude of both of these scattering processes are proportional to the e-e interaction strength inside the QW . The competition between these two scattering processes which actually arise due the same e-e interaction strength inside the QW leads to the presence of the new SFP where all the scattering amplitudes have intermediate non-zero values. This fact is unique about this fixed point and hence this fixed point is the central focus of our discussion. Details of this fixed point are further elaborated in the discussion at the end of this article.

We adopt the same procedure as described above for the three-wire junction but now with $SU(4)$ generators. This is so because the full 8×8 $\mathbb{S}$-matrix describing the NSN junction has a block diagonal form with each spin block (up and down spin sectors) being represented by a 4×4 matrix. Hence we have a unitary starting $\mathbb{S}$-matrix deviating from the fixed point $\mathbb{S}$-matrix ($\mathbb{S}_0$), as given before by Eq. 6, except that now the sum over j runs from 1 to 16 since λ_j's now represent the fifteen generators of the $SU(4)$ group along with the identity matrix. The $\mathbb{S}_0$ which describes the SFP[23] is given by $r = 1/2$, $t = 1/2$, $r_A = -1/2$ and $t_A = 1/2$. Note that the SFP is a particle-hole, left-right symmetric fixed point and hence the entire 4×4 $\mathbb{S}$-matrix is determined completely by the above given four amplitudes for r, t, r_A, t_A.

We then solve Eq. 8 for this case with sixteen coupled equations up to the first order in the small perturbations ϵ_j's. We obtain the sixteen eigenvalues which correspond to the power laws around the sixteen eigen-directions. These power laws around the various eigen-directions can be listed as [0, 0, 0, 0, 0, $-\alpha/2$, $-\alpha/2$, $(\alpha - \alpha')/2$, $\alpha'/2$, $\alpha'/2$, $(-\alpha + \alpha')/2$, $(-\alpha + \alpha')/2$, $(\alpha+\alpha')/2$, $(\alpha+\alpha')/2$, $(\alpha-\alpha'-\sqrt{9\alpha^2 + 14\alpha\alpha' + 9\alpha'^2})/4$, $(\alpha - \alpha' + \sqrt{9\alpha^2 + 14\alpha\alpha' + 9\alpha'^2})/4$].
Hence we note that there are five marginal directions, two stable directions, four unstable directions and four other directions whose stability depends on the sign of $\alpha - \alpha'$. One of the most striking outcomes of this analysis is the fact that we obtain two power laws which are not just simple linear combinations of $V(0)$ and $V(2k_F)$. Instead, they appear as square roots of quadratic sum of these quantities. Our analysis actually leads to the first demonstration of the existence of such power laws in the context of quantum impurity problems in LL theory and this is the central result of this article.

Having obtained the power laws the next task is to obtain an explicit expression for the Landauer-Büttiker conductance corresponding to perturbations around these fixed points along some of the eigen-directions. Now note that the RG equation is expressed in terms of ϵ''s whereas the $\mathbb{S}$-matrix representing small deviations from the fixed point is expressed in terms of ϵ's. The two terminal linear conductance across the junction depends explicitly on the $\mathbb{S}$-matrix element which are expressed in terms of ϵ's (see Eq. 6). Hence in order to obtain an expression for conductance in terms of the temperature or the applied voltage dependence induced by e-e interaction, we need to first assign bare values to the various perturbations parameterized by ϵ's and then express the ϵ''s evolved under RG flow in terms of these bare values of ϵ''s as $\epsilon'(\Lambda) = (\Lambda/\Lambda_0)^\mu \epsilon'_0$ where Λ corresponds to the energy scale at which we are probing the system (which can be either voltage bias at zero temperature or temperature at vanishing bias) and Λ_0 is the high energy cut-off expressed in terms of voltage or temperature. Then by using the rotation matrix which diagonalizes the coupled RG equations, we express ϵ's in terms of ϵ''s written explicitly as a function of temperature or voltage. Finally plugging these renormalized values of ϵ's into the $\mathbb{S}$-matrix given by Eq. 6, we get all the transmission and reflection amplitudes for the system as explicit functions of the temperature or voltage carrying the specific power laws associated with perturbations switched on along the eigen-directions. These amplitudes are now directly related to the linear conductances.
Now we will calculate expression for conductance for a simple case where only one of the $\epsilon'(= \epsilon'_{15})$ is turned on. For this we need the $\mathbb{U}$ matrix for this case which is given by

$$\mathbb{U} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \sqrt{3/2} & 0 & 0 & 0 & 0 & -1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\sqrt{3/2} & 0 & 0 & -\sqrt{3/2} & 0 & -1/\sqrt{2} & \sqrt{3/2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 & 0 & 1 & 0 & \sqrt{3} & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & A & 0 & 1 & 0 & 0 & 1 & 0 & A & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & B & 0 & 1 & 0 & 0 & 1 & 0 & B & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \tag{11}$$

where, $A = 4(\alpha + \alpha')/(\alpha - \alpha' - \sqrt{9\alpha^2 + 14\alpha\alpha' + 9\alpha'^2})/4$ and $B = 4(\alpha + \alpha')/(\alpha - \alpha' + \sqrt{9\alpha^2 + 14\alpha\alpha' + 9\alpha'^2})/4$.

We choose this specific direction to perturb the system as this corresponds to a power law which is not a linear function of $V(0)$ and $V(2K_F)$ and hence interesting to study. The $\mathbb{S}$-matrix to quadratic order in ϵ'_{15} is given by

$$\mathbb{S} = \begin{bmatrix} \frac{[1-(1-i)\epsilon'_{15}-{\epsilon'_{15}}^2]}{2} & \frac{[1+\epsilon'_{15}]}{2} - \frac{[1-i]{\epsilon'_{15}}^2}{4} & -\frac{1}{2} & \frac{[1+i\epsilon'_{15}]}{2} - \frac{[1+i]{\epsilon'_{15}}^2}{4} \\ \frac{[1-(1+i)\epsilon'_{15}-{\epsilon'_{15}}^2]}{2} & \frac{[1+\epsilon'_{15}]}{2} - \frac{[1+i]{\epsilon'_{15}}^2}{4} & \frac{1}{2} & \frac{[1-i\epsilon'_{15}]}{2} + \frac{[1-i]{\epsilon'_{15}}^2}{4} \\ -\frac{[1+(1-i)\epsilon'_{15}-{\epsilon'_{15}}^2]}{2} & \frac{[1-\epsilon'_{15}]}{2} - \frac{[1-i]{\epsilon'_{15}}^2}{4} & \frac{1}{2} & \frac{[1-i\epsilon'_{15}]}{2} - \frac{[1-i]{\epsilon'_{15}}^2}{4} \\ \frac{[1+(1+i)\epsilon'_{15}-{\epsilon'_{15}}^2]}{2} & -\frac{[1-\epsilon'_{15}]}{2} + \frac{[1+i]{\epsilon'_{15}}^2}{4} & \frac{1}{2} & \frac{[1+i\epsilon'_{15}]}{2} + \frac{[1-i]{\epsilon'_{15}}^2}{4} \end{bmatrix} \quad (12)$$

So, the scaling of sub-gap conductance (to $\mathcal{O}(\epsilon'^2_{15})$) for an incident electron and a hole taking into account both spin-up and spin-down contributions in units of $2e^2/h$ is given by

$$G^e_{12} = -\frac{\epsilon'_{15}}{2}\ ; \quad G^e_{21} = \frac{\epsilon'_{15}}{2} \quad (13)$$

$$G^h_{12} = -\frac{\epsilon'_{15}}{2}\ ; \quad G^h_{21} = \frac{\epsilon'_{15}}{2} \quad (14)$$

where $\epsilon'_{15} = \epsilon'_{15,0}(\Lambda/\Lambda_0)^{(\alpha-\alpha'-\sqrt{9\alpha^2+14\alpha\alpha'+9\alpha'^2})/4}$. Here the superscripts e and h stand for electrons and holes while the subscripts 1 and 2 stand for first and second wire respectively. Also, $G^e_{12} = |t^{eh}_{A,12}|^2 - |t^{ee}_{12}|^2$ where t^{ee} is the transmission amplitude for electrons and t^{eh}_A represents CAR amplitude for electrons. Similar expressions hold for the holes. In the expressions of power laws given above, $\alpha = (g_2 - 2g_1)/2\pi\hbar v_F$ and $\alpha' = (g_1 + g_2)/2\pi\hbar v_F$ where the bare values of $g_1(d) = V(2k_F)$ and $g_2(d) = V(0)$. In our stability analysis, we have assumed $\alpha < \alpha'$ which is consistent with experimental observations[38]. For the special case when $g_2 = 2g_1$, α vanishes and only α' survives.

It is very interesting to note that even though the $\mathbb{S}$-matrix corresponding to perturbation along ϵ'_{15} breaks both time reversal and electron-hole symmetry, the two terminal linear conductance restores particle-hole symmetry. Secondly it might be of interest to note the fact that the fixed point conductance admits correction along ϵ'_{15} which is linear in ϵ'_{15} and not quadratic. Normally when we perform a stability analysis around a fixed point $\mathbb{S}$-matrix whose elements are constituted out of unimodular numbers (representing disconnected or perfectly connected fixed points), it is always possible to identify various terms of the $\mathbb{S}$-matrix, representing small unitary deviations from the fixed point $\mathbb{S}_0$-matrix in terms of various tunneling operators which are perturbatively turned on around the fixed point Hamiltonian. Hence a straight forward perturbative linear conductance calculation using the Hamiltonian along with the tunneling parts will suggest that the correction due to the $\mathbb{S}$-matrix representing small deviation from fixed point $\mathbb{S}_0$-matrix must introduce correction to fixed point conductance which are quadratic in terms of the deviation parameter. But this argument applies only to those fixed points which correspond to completely connected or disconnected wires and not to fixed points which have intermediate values for various transmission and reflection amplitudes like the SFP . In other words, an arbitrary deviation from SFP may not be easily representable as a tunneling operator. This explains why the linear dependence of the conductance on ϵ' and hence the corresponding power law dependence looks unconventional.

As a cross check, we see that we get back the power laws associated with the symmetric fixed point[4] of the four-wire junction once we substitute $\alpha' = 0$ in the expression for the power laws of the SFP for the NSN junction. Although our geometry does not correspond to the real junction of four LL wires, the presence of both electron and hole channel mimics the situation of a four-wire junction. More specifically, the symmetric fixed point of the NSN junction (SFP) turns out to be identical to the symmetric fixed point of the four-wire junction due to perfect particle-hole symmetry of the SFP when α' is set to zero.

Next we enumerate and discuss the stability of the other fixed points (RFP, AFP, TFP and CAFP) obtained in Ref. 23 for the NSN junction :

(a) $t = t_A = r_A = 0, r = 1$ (RFP) : This fixed point turns out to be stable against perturbations in all directions. There are ten directions for which the exponent is -α while two others with the exponents -(α+α'). The remaining four directions are marginal.

(b) $t = t_A = r = 0, r_A = 1$ (AFP) : This is unstable against perturbations in twelve directions. There are ten directions with exponent α and two directions with exponent (α+α'). The remaining four directions are marginal, as for RFP.

(c) $r_A = t_A = r = 0, t = 1$ (TFP) : This fixed point has four unstable directions with exponent α, two stable directions with the exponent -α' and the remaining directions are marginal.

(d) $r_A = t = r = 0, t_A = 1$ (CAFP) : This has four unstable directions with exponent α and two stable directions with the exponent -α' and the remaining directions are marginal.

Note the close similarity in stability between CAFP and TFP fixed points. This can be attributed to the fact that both these fixed points belong to the continuous family of marginal fixed points defined by the condition $|t|^2 + |t_A|^2 = 1$. The entire family of fixed points is marginal because for these fixed points, the amplitudes for Friedel oscillation and pair potential in the wire vanish identically.

Hence, we notice that for the AFP only the scattering amplitude from the pair potential inside the QW is nonzero as only r_A is nonzero, and for RFP only the scattering amplitude from Friedel oscillations are non-zero as only r is nonzero. Furthermore, both for CAFP and TFP , the amplitude for scattering from the Friedel oscillations as well as from the pair potential is zero as in these cases both r and r_A are zero. So SFP is the only fixed point for which both the amplitude for scattering from the Friedel oscillations and the pair amplitude are finite; hence, this fixed point is nontrivial. Its very existence can be attributed to the interplay of these two different scattering processes arising from Friedel oscillations and the pair potential inside the wire. The conductance at this fixed point gets contribution from both the elastic co-tunneling (CT) of electrons through the superconductor as well as through the crossed Andreev reflection (CAR) process. Since both electron and hole channels contribute with opposite signs to conductance, if we give a small perturbation around this fixed point, we get an interesting non-monotonic behavior of the conductance $G_{\mathsf{NSN}} = G_{\mathsf{CAR}} - G_{\mathsf{CT}}$. This effect emerges due to the competition between the electron and the hole channel and it can be of interest from an experimental point of view. Also note that at the SFP , CT amplitude of electrons $t = 1/2$ and the CAR amplitude $t_A = 1/2$. This means that if we have an incident spin-polarized beam of (say "up" polarized) electrons on the junction, when the junction is tuned to this fixed point, 25% of the spin up electrons get transmitted through the junction and 25% of the spin up electrons get converted to spin up holes as they pass through the junction. Hence the transmitted charge across the junction is zero on the average, but there is pure spin current flowing out of the junction. Equivalently, we can think that the pure spin current is generated due to flow of two beams of electrons of equal intensity, one with spin up electrons and the other with spin down electrons propagating in opposite directions. Therefore the SFP can be relevant for future spintronics applications. These points have been discussed in detail in Ref. 22.

To summarize, we have laid down a scheme to perform a systematic stability analysis which works well for both normal and superconducting junctions of multiple LL QW. Using our procedure, we reproduced the known power laws for the fixed points of the three- and four-wire junctions. Then we applied it to the NSN junction and established the existence of non-trivial power laws which are non-linear functions of $V(0)$ and $V(2k_F)$. Finally, we calculated the Landauer-Büttiker conductance associated with the perturbations switched on around these fixed points and found the explicit voltage or temperature power law dependence.

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