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Non Possum Comprimi Ergo Sum: Skyrmions and Edge States in the Quantum Hall Effect

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Abstract

When the chemical potential of an electron system has a discontinuity at a density n^*, the system is said to be incompressible and a finite energy is required to create mobile charges in the bulk of the system. The quantum Hall effect is associated with incompressibilities in a two-dimensional electron system that occur at magnetic-field dependent densities, $n^*(B)$. In these notes we discuss two aspects of the physics of quantum Hall systems that follow directly from this association.

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I. INTRODUCTION

The quantum Hall effect [1,2] is an anomaly that occurs in the transport properties of two-dimensional electron system (2DES) in the regime of strong perpendicular magnetic fields. At certain magnetic fields it is found that the voltage drop in the system in the direction of current flow, which is responsible for dissipation, vanishes at low temperatures. Our understanding of this transport anomaly is not absolutely complete, however there is fairly broad agreement that the effect can occur only when the electronic system has, in the absence of disorder, jumps in its chemical potential at certain densities (n^*), that depend on magnetic field. In Section II of these notes we specifically discuss the relationship between incompressibility and transport properties in a 2DES. The remaining sections discuss two aspects of the physics of quantum Hall systems that are direct consequences of this relationship.

Quantum Hall systems can be ferromagnetic. Under appropriate circumstances the spin-moments of the electrons align spontaneously, *i.e.* in the absence of Zeeman coupling to a magnetic field. We will refer to the 2DES in this case as a quantum Hall ferromagnet. Quantum Hall ferromagnets provide a particularly simple example of a two-dimensional itinerant electron ferromagnet, and therefore represent an attractive target for theories of the quantum statistical mechanics of such systems [3–5]. They also have a number of interesting unique properties, the most striking of which is that their instantons carry an electronic charge. This property is an immediate consequence of *incompressibility at a magnetic-field dependent density*. In Section III of these notes we discuss the electrically charged instantons of quantum Hall ferromagnets and some of the observable consequences of their presence.

Another consequence of incompressibility at a magnetic field dependent density is that quantum Hall systems *necessarily* have gapless excitations localized at their edges. The low-energy physics of quantum Hall systems is therefore like that of a one-dimensional electron system which, because of time-reversal symmetry breaking by the magnetic field, can carry a current even in equilibrium. In Section IV we will discuss the physics of quantum Hall

edges, including their description in terms of the Luttinger liquid models appropriate to one-dimensional fermion systems. Section V contains some concluding remarks.

These notes do not attempt a systematic introduction to the physics of the quantum Hall effect. Interested readers can consult a previous effort of mine [6] that is cited frequently below. If greater depth is desired, readers can consult one of the excellent books covering different aspects of the subject [2]. I have written previously at this level but at greater length on quantum Hall edge states [7]; parts of the present notes borrow from that text.

II. INCOMPRESSIBILITY AND THE QUANTUM HALL EFFECT

The thermodynamic compressibility of a system of interacting particles is proportional to the derivative of the chemical potential with respect to density. It can happen that at zero temperature the chemical potential has a discontinuity at a density n^*: the energy to add a particle to the system (μ^+) differs, at this density, from the energy to remove a particle from the system (μ^-). The system is then said to be incompressible. In an incompressible system a finite energy is required to create unbound positive and negative charges that are capable of carrying current through the bulk. The number of these free charges present in the system will have an activated temperature dependence and will vanish for $T \to 0$. Incompressible systems are usually insulating. Paradoxically, as we explain below, incompressibility is precisely the condition required for the quantum Hall effect to occur. The twist is that in the case of the quantum Hall effect, the density n^* at which the incompressibility occurs must depend on magnetic field. In my view, *incompressibility at a magnetic-field-dependent density* is the *sine qua non* of the quantum Hall effect.

For non-interacting electrons, the single-particle energy spectrum of a 2DES in a magnetic field consists of Landau levels, separated by $\hbar\omega_c \equiv \hbar eB/m^*c$ energy gaps and with a macroscopic degeneracy $N_\phi = AB/\Phi_0 = A/(2\pi\ell^2)$. (Here B is the magnetic field strength, $\Phi_0 = hc/e$ is the magnetic flux quantum, A is the area of the system, and m^* is the electron mass.) Both the energy gaps between Landau levels and the degeneracy of the Landau levels

are proportional to B. Chemical potential discontinuities occur whenever the density is an integral multiple of $n^* = B/\Phi_0$. We show below that this property requires the existence of the gapless edge excitations discussed in Section IV. It is conventional in discussing the quantum Hall effect to use a magnetic field dependent density unit by defining the Landau level filling factor $\nu \equiv n/n^* = 2\pi\ell^2 n$. For non-interacting electrons, incompressibilities occur at integer filling factors. When interactions are included, incompressibilities can also occur at fractional filling factors and, as we will discuss, physical properties near integer filling factors can be qualitatively altered.

The relationship between incompressibility and the transport anomalies that give the quantum Hall effect its name can be understood by the following argument [8]. Consider a 2DES at zero temperature, as illustrated in Fig. [1]. We consider the case in which the chemical potential lies in the 'charge gap'; $\mu \in (\mu^-, \mu^+)$. We want to consider the change in the equilibrium local currents, present in the system because of the breaking of time-reversal-invariance by the magnetic field, when we make an infinitesimal change in the chemical potential, $\delta\mu$. Because μ lies in the charge gap the change in the local current density anywhere in the bulk of the system must be zero. The current density can change, if it does anywhere, only at the edge of the system. It follows from charge conservation that, if there is a change in the current flowing along the edge of the system, it must be the same at any point along the edge. We can relate this change in current to the change in the orbital magnetization:

$$\delta I = \frac{c}{A}\delta M. \tag{1}$$

Eq. (1) is just the equation for the magnetic moment of a current loop. However,

$$\delta M = \frac{\partial M}{\partial \mu}|_B \, \delta\mu = \frac{\partial N}{\partial B}|_\mu \, \delta\mu. \tag{2}$$

(N is the number of electrons.) The second equality in Eq. (2) follows from a Maxwell relation. Combining Eq. (1) and Eq. (2) we obtain the following result for the rate at which the equilibrium edge current changes with chemical potential when the chemical potential lies in a charge gap:

$$\frac{\delta I}{\delta \mu} = c \frac{\partial n^*}{\partial B}|_{\mu}. \tag{3}$$

The fact that $\delta I/\delta\mu \neq 0$ implies that whenever the charge gap occurs at a density that depends on magnetic field, there *must* be gapless excitations at the edge of the system. Properties of these low-energy edge excitations are discussed in Section IV.

Eq.(3) is expected to apply to the edge states even when the chemical potential lies only in a mobility gap and not in a true gap, as illustrated schematically in Fig. [1]. A net current can be carried from source to drain across the system by changing the local chemical potentials only at the edges and having different chemical potentials along the two edges connecting source and drain. When bulk states are localized, the two edges and the bulk are effectively decoupled from each other. Eq. (3) then also applies to transport currents, relating the current carried from source to drain to the chemical potential difference between the two edges, equal to eV_H where V_H is the Hall voltage. There is no voltage drop along an edge since each edge is in local equilibrium and hence no dissipation inside the sample. Eq. (3) is commonly known as the Středa-Widom formula [9]. In using this picture to explain transport experiments in bulk systems it is necessary to claim that the transport current will be carried entirely at the edge of the system even when bulk states occur at the Fermi level, as long as these states are localized. There are difficulties with this argument as a complete explanation for all transport phenomena associated with the quantum Hall effect, but that is another story and we will not pursue it here.

III. QUANTUM HALL FERROMAGNETS

A. Energy Scales

The quantum Hall regime is usually understood as the regime in which no qualitative change in physical properties results from mixing of Landau levels by either interactions or disorder. It is common in theoretical studies to truncate the Hilbert space to a single orbital Landau level and include mixing , if at all, only when making quantitative estimates for

comparison with experiment. The quantum Hall regime, then, assumes that the Landau level separation, $\hbar\omega_c$ is larger than other energy scales of interest. On the other hand the ferromagnetic state is generally defined in terms of the properties of an electronic system in the absence of a magnetic field. The term *Quantum Hall Ferromagnet* appears to be an oxymoron. To understand why it is not only sensible but also of more than academic interest to add this category to our taxonomy of electronic states, it is necessary to consider the relevant energy scales for the case of the semiconductor systems in which 2DES's are realized. For a free-electron system in a magnetic field, the Zeeman splitting of spin-levels $g\mu_B B$ and the Landau level separation $\hbar\omega_c$ are identical, apart from small relativistic corrections. Electrons in states near the conduction band minimum of a semiconductor behave like free electrons [10] except that band effects renormalize the electron mass m^* and the g-factor. In the case of the GaAs systems, where the quantum Hall effect is most often studied, band effects increase the Landau level separation by a factor of ~ 20 and reduce the Zeeman splitting by a factor of ~ 4. As a result for typical experimental situations, the Landau level separation (in temperature units) is ≈ 200K, and the characteristic scale for electron-electron interactions is ≈ 100K while the Zeeman splitting is only ≈ 2K. We call a system a quantum Hall ferromagnet if the electronic spins in the incompressible ground state with density n^* align in the absence of Zeeman coupling. In many cases, the properties of a quantum Hall ferromagnet with such a small Zeeman coupling do not differ noticeably from the properties when the Zeeman coupling is set to zero. In other cases, the small Zeeman coupling plays an important role but it is still useful to treat the system as a ferromagnet in the presence of a small symmetry breaking field.

B. Ferromagnetic Ground States

The Hartree-Fock approximation, in which many-electron states are approximated by single Slater determinants, provides a simple explanation for itinerant electron ferromagnetism that is often qualitatively correct. For non-interacting electrons, energy is minimized

by occupying both spin-states of each single particle energy level. (If the number of majority-spin electrons exceeds the number of minority-spin electrons it is necessary to occupy higher energy single-particle levels.) The ground state thus has equal numbers of majority-spin and minority-spin electrons if the total number of electrons is even and the difference is one if the total number of electrons is odd. This statement applies for an arbitrary spin-quantization axis. As long as there is no spin-orbit coupling, the Hamiltonian is invariant under global spin rotations and the total spin of all electrons is a good quantum number. The above statement is equivalent to the observation that for non-interacting electrons the ground state always has total spin $S = 0$ if the number of electrons is even and total spin $S = 1/2$ if the number of electrons is odd. However, interaction energies are lower in single-Slater-determinant states with higher values of the total spin and, generally, is minimized in fully spin-polarized states with $S = N/2$. As in the familiar Hund's rules from atomic physics, higher spin states tend to have lower interaction energies because like-spin electrons are prevented from being at the same position by the Pauli exclusion principle and therefore have more energetically favorable spatial correlations. In the Hartree-Fock approximation, or in closely related spin-density-functional approximations, an itinerant electron system is expected to be ferromagnetic if the reduction in interaction energy due to creating a finite spin-polarization state exceeds the cost in single-particle kinetic energy, or more generally 'band' energy. Because of Landau level degeneracy, the cost in kinetic energy of creating a finite spin polarization for electrons in a magnetic field is precisely zero unless ν is an even integer. Hartree-Fock or similar approximations would predict a ferromagnetic ground state for electrons at nearly any value of ν. In fact, this conclusion is incorrect. For example, at certain filling factors it is known [11] that the interaction energy is minimized in a $S = 0$ state. We do believe, however, that there exist finite ranges of filling factor over which the ground state has $S/N \neq 0$. For a deeper understanding of this behavior, we need a more rigorous argument.

The approach we now describe is in the same spirit as the illuminating outlook on the spin-polarized fractional quantum Hall effect that arises from appropriate hard-core model

Hamiltonians [12,13]. As discussed for the case of interest below, these models have zero energy many-particle eigenstates that are often known analytically, are separated from other many-particle states by a finite gap, and have a degeneracy that increases with decreasing N. The incompressible state responsible [6] for a quantum Hall effect transport anomaly in such a model is the nondegenerate maximum N zero energy eigenstate. The zero energy eigenstates at lower densities constitute the portion of the spectrum that involves only the degrees of freedom of the, in general fractionally charged [14], quasiholes of the incompressible state. It is assumed that the difference between the model Hamiltonian and the true Hamiltonian is a sufficiently weak perturbation that the quasihole states are still well separated from other states in the Hilbert space, although accidental degeneracies will be lifted in the spectrum of the true Hamiltonian. Here we apply this approach to argue that the ground state at $\nu = 1$ is a quantum Hall ferromagnet with $S = N/2$.

For our analysis we use the symmetric gauge in which the single-particle orbitals [6] in the lowest Landau level are

$$\phi_m(z) = \frac{z^m}{(2^{m+1}\pi m!)^{1/2}} \exp(-|z|^2/4), \tag{4}$$

where [15] $m = 0, 1, \cdots, N_\phi - 1$, $z = x + iy$, and x and y are the Cartesian components of the two-dimensional coordinate. We study here a hard-core model for which the interaction is:

$$V = 4\pi V_0 \sum_{i<j} \delta^{(2)}(\vec{r}_i - \vec{r}_j) \tag{5}$$

At strong magnetic fields the low-energy Hamiltonian is simply the projection of this interaction onto the lowest Landau level [12]. Many-particle wavefunctions that are zero energy eigenstates of this Hamiltonian must vanish when any two-particles are at the same position and must therefore have the difference coordinate for each pair of particles as a factor:

$$\Psi[z, \chi] = [\prod_{i<j}(z_i - z_j)]\, \Psi_B[z, \chi]. \tag{6}$$

We note that the each complex coordinate appears to the power $N - 1$ in the factor in square brackets in Eq. (6) and that this factor is completely antisymmetric. It follows that

$\Psi_B[z]$ must be a wavefunction for N *bosons* and that these bosons can be in states with angular momenta from 0 to $N_\phi - N$. This simple observation leads to the conclusions we reach below.

In these notes we discuss only the case where $N = N_\phi$; the same approach can [16] be extended to $N \neq N_\phi$ to elucidate the physics of charged excitations in quantum Hall ferromagnets but here we will follow another line for this part of the discourse. For $N = N_\phi$ all bosons must be in orbitals with $m = 0$. $\Psi_B[z, \chi]$ must then be proportional to a symmetric many-particle spinor and therefore have total spin quantum number $S = N/2$. The orbital part of the fermion wavefunction can be recognized as the Slater determinant with all orbitals from $m = 0$ to $m = N_\phi - 1$ occupied. We are able to conclude that the ground state is a strong ferromagnet with no orbital degeneracy. The ease with which this conclusion can be reached contrasts markedly with the case of the Hubbard model where enormous effort has yielded relatively few firm results [17]. When Zeeman coupling is included in the Hamiltonian, the ground state will be the member of this $N + 1$ fold degenerate multiplet for which all spins are aligned with the magnetic field, *i.e.* the state with $S_z = S = N/2$.

C. Charged Instantons

In this section we follow a line of argument that emphasizes the role of *incompressibility at a magnetic field dependent density* in the unusual properties of quantum Hall ferromagnets. For the most part we follow Sondhi *et al.*, in applying the non-linear σ model ($NL\sigma$) field-theoretical description of a ferromagnet to the present case [18]. The $NL\sigma$ model is intended to capture the long-wavelength low-energy physics of isotropic ferromagnets. In the $NL\sigma$ model, the energy is expressed as a functional of a unit vector $\hat{m}(\vec{r})$, which specifies the direction of the ordered spin-moment as a function of position:

$$E = E_0 + \frac{\rho_s}{2} \int d^2\vec{r} |\nabla \hat{m}(\vec{r})|^2. \tag{7}$$

We will refer to a particular configuration of the ferromagnet specified by a function $\hat{m}(\vec{r})$ as a spin-texture. Here E_0 is the ground state energy, which is independent of the direction

of the ordered moment as long as it is constant in space, and ρ_s, the spin-stiffness, is a phenomenological constant that must be determined from experiment or calculated from a microscopic model. The low-energy long-wavelength physics of ferromagnets is dominated by variations in the direction of the ordered moment that are slow on microscopic length scales and that can cost vanishingly small energies.

An important aspect of any field theory is the enumeration of its instantons [19]. Instantons are finite excitation energy extrema of an energy (or action) functional, in which the variation of the field is localized in space. Typically, the stability of an instanton is associated with a topological classification of field configurations. For $NL\sigma$ models in two space dimensions and with three-dimensional unit vector fields, all finite energy spin configuration can be classified by an integer valued topological index, sometimes called a topological charge, which specifies the number of times the order parameter field is wrapped around the unit sphere when the position $\vec{r}$ is varied over two-dimensional space. Field configurations with different topological indices cannot be continuously deformed into one another. The sign of the topological index depends on the sense of the closed paths traced out on the surface of the unit sphere when $\vec{r}$ traces a closed path in space. An explicit expression for the topological index, $Q[\vec{m}]$, associated with a spin-texture can be derived by first calculating the solid angle enclosed on the sphere by $\hat{m}(\vec{r})$ when $\vec{r}$ encloses an infinitesimal area element in space, and then integrating over space:

$$Q[\vec{m}] = \frac{1}{4\pi} \int d^2\,\vec{r} \hat{m}(\vec{r}) \cdot [\partial_x \hat{m}(\vec{r}) \times \partial_y \hat{m}(\vec{r})]. \tag{8}$$

Eq.[8] follows from the observation that for a unit vector field, $\partial_i \hat{m}(\vec{r})$ is orthogonal to $\hat{m}(\vec{r})$.

For the 2D $NL\sigma$ model it is possible to derive analytic expressions for the lowest energy spin-textures of a given topological charge [19,20]. The lowest energy textures with unit magnitude topological charge, commonly called Skyrmions, have a spin-texture of the following form:

$$m_x = 2x\lambda/(r^2 + \lambda^2)$$

$$m_y = \pm 2y\lambda/(r^2 + \lambda^2)$$

$$m_z = (r^2 - \lambda^2)/(r^2 + \lambda^2) \tag{9}$$

Here λ is an arbitrary length scale, $\hat{m}$ has an arbitrary global orientation fixed here by setting $\hat{m} = \hat{z}$ for $r \to \infty$, and the skyrmion is centered at an arbitrary point chosen as the origin of the coordinate system. It is easy to verify that these spin textures have topological charge $Q = \pm 1$ and excitation energy $E - E_0 = 4\pi\rho_s$. The form of the Skyrmion spin texture is illustrated in Fig.[2].

The $NL\sigma$ model considerations in the above paragraph are appropriate for any 2D ferromagnet. Skyrmions lead to contributions to the physical properties of 2D ferromagnets that have a non-analytic temperature dependence of the form $\exp(-4\pi\rho_s/k_BT)$ at low temperatures. These non-analytic terms are interesting, since they cannot be captured by perturbative theories, but they typically produce only subtle corrections to a temperature dependence that is dominantly controlled by thermally excited spin-wave excitations. In the case of quantum Hall ferromagnets, however, Skyrmions carry an electrical charge and as a consequence play a more prominent role in determining the physical properties of the system. To understand why Skyrmions carry a charge it is useful to consider first the case of a non-interacting electron in the presence of a magnetic field that couples to its orbital degrees of freedom and an independent strong Zeeman field that couples to its spin and whose orientation, specified by $\hat{m}(\vec{r})$, varies slowly in space. The direction of the electron spin will vary in space to maintain alignment with the external magnetic field. As is familiar from the calculation of spin Berry phases, the changing spin direction changes the adiabatic orbital Hamiltonian that now [21] takes the form:

$$H = \frac{1}{2m}(\vec{p} + \frac{e}{c}(\vec{A} + \vec{A}_B))^2. \tag{10}$$

Here the effective magnetic field due to the varying spin orientation,

$$\nabla \times \vec{A}_B = \frac{\Phi_0}{4\pi}\hat{m} \cdot [\partial_x \hat{m}(\vec{r}) \times \partial_y \hat{m}(\vec{r})], \tag{11}$$

is proportional to the topological index density of the unit vector field $\hat{m}(\vec{r})$. In a Hartree-Fock approximation, the exchange interaction will produce a strong effective magnetic field

that points in the same direction as the local order parameter, so we can associate $\hat{m}(\vec{r})$ above with the order parameter field. It follows that the effective value of N_ϕ is changed by one when the order parameter has a texture with unit topological index. Moreover, as we have emphasized in these notes, the charge gap of the quantum Hall effect occurs at an electron density that depends on magnetic field. For a quantum Hall effect at Landau level filling factor ν we conclude that the electron number at which the charge gap occurs changes by $\nu Q[\hat{m}]$ when the order parameter field has topological index $Q[\hat{m}]$. The same conclusion can be reached [18,22] using the Chern-Simons Landau-Ginzburg theory of the quantum Hall effect [23,24] or by explicit calculation [25].

The Skyrmion excitations of quantum Hall ferromagnets are responsible for striking physical effects because they carry a physical charge and *are present in the ground state* for filling factors near those at which the incompressible state occurs. To be concrete, we consider the quantum Hall ferromagnet that occurs at $\nu = 1$. For $N = N_\phi$ the ground state has $S = N/2$ as discussed above. For $N = N_\phi \pm 1$, the ground state contains a single charged Skyrmion. The Skyrmion can be introduced by changing the total electron number or, in what is the typical experimental situation, by changing the magnetic field strength and hence N_ϕ. In the $NL\sigma$ model the energy of a Skyrmion is independent of its size. To represent Skyrmions in quantum Hall ferromagnets, however, it is necessary to add additional terms to the model to account for Zeeman coupling and for the Coulomb self-interaction energy of a Skyrmion. Zeeman coupling favors small Skyrmions, since the spin near the center of the Skyrmion is oriented in opposition to the Zeeman field. On the other hand the repulsive Coulomb energy favors large Skyrmions. For typical experimental situations the optimal Skyrmion size is not so much larger than microscopic lengths, invalidating the use of the field-theory description for quantitative estimates. In a quantum description, the number of reversed spins per skyrmion is quantized so that, when Skyrmion-Skyrmion interactions can be neglected, we expect that the component of the total spin along the direction of the Zeeman field is

$$S_z = N/2 - (K + \theta)|N - N_\phi| \tag{12}$$

Here $\theta = 1$ for $N > N_\phi$ and $\theta = 0$ for $N < N_\phi$. $|N - N_\phi|$ is the number of Skyrmions or antiskyrmions present in the system. The integer quantum number K will depend in general on the relative size of Zeeman and Coulomb interaction terms and is the relevant quantum measure of the Skyrmion size. For non-interacting electrons, or with interactions treated in the Hartree-Fock approximation, $K = 0$ so that S_z always has the maximum value allowed by the Pauli exclusion principle. (K is guaranteed by particle-hole symmetry [26] to have the same value for $N > N_\phi$ and $N < N_\phi$.) The $NL\sigma$ model considerations of Sondhi et al. [18] described above, and also earlier numerical exact diagonalization calculations [27], suggest that K should be non-zero for quantum Hall ferromagnets and quite large if the Zeeman energy is small. These predictions were dramatically confirmed when Barrett *et al.* unexpectedly succeeded [28] in using optical pumping techniques to perform NMR Knight shift measurements of the spin-polarization of two-dimensional electron systems in the quantum Hall regime. The results of this experiment are illustrated in Fig. 3 and correspond to $K = 3$, in quantitative agreement with microscopic predictions based on a generalized Hartree-Fock approximation for single-Skyrmion states [29]. There seems to be little doubt that the elementary charged excitations of quantum Hall ferromagnets are Skyrmion-like objects that carry large spin quantum numbers. Recent transport [30] and optical [31] experiments add additional support to this conclusion.

For large enough $|N - N_\phi|$ the Skyrmion-like objects will eventually interact strongly. When the density of Skyrmions is low and the temperature is low, Skyrmions are expected to form a lattice similar to the Wigner crystal state formed by electrons in the limit of very strong magnetic fields. In Fig. 4 we compare theoretical calculations of the spin-polarizations as a function of filling factor for several candidate Skyrmion lattice states with experimental data. The theoretical results were obtained by Brey *et al.* using a generalized Hartree-Fock approximation [32] and illustrate several important aspects of the physics of Skyrme crystals. These authors find that the ground state of the Skyrme crystal is a square lattice rather than

a triangular lattice as found for the electron Wigner crystal. Furthermore, as illustrated in Fig. 4, the spin-polarization of the square lattice Skyrme crystal is much smaller, for a given Zeeman coupling strength, than for the lowest energy triangular lattice state. The preference for a square lattice can be understood qualitatively in terms of the $NL\sigma$ model description of Skyrmion states. For that model Skyrmions are centered at an arbitrary point, have an arbitrary size, and are invariant under arbitrary global spin rotations. When Coulomb and Zeeman energies are included the optimal Skyrmion size is fixed and the spin moment must be aligned with the Zeeman field far from the Skyrmion center. However, the energy of each Skyrmion is still invariant under global rotations of the moment about an axis aligned with the Zeeman field. For a Skyrme lattice the relative values of these rotation angles must be adjusted to minimize the total energy. It turns out that the interaction energy between a pair of Skyrmions is reduced when they have opposing orientations for the component of the ordered moment perpendicular to the Zeeman field. This arrangement allows the ordered moment orientation to vary more smoothly along the line connecting Skyrmion centers. The tendency toward opposing orientations is frustrated on triangular lattice, hence the energetic preference for a square lattice. The stronger short-range repulsive interaction in the aligned orientation ferromagnetic lattice case, results in smaller Skyrmions and therefore more spin-polarized states. The spin-polarizations calculated for the opposing orientation, square lattice case shown in Fig. 4 appear to be in excellent agreement with experiment over a wide range of filling factors near $\nu = 1$.

IV. EDGE EXCITATIONS OF AN INCOMPRESSIBLE QUANTUM HALL FLUID

A. Non-Interacting Electron Picture

Throughout this section we will consider a disk geometry where electrons are confined to a finite area centered on the origin by a circularly symmetric confining potential, $V_{\rm conf}(r)$. We have in mind the situation where $V_{\rm conf}(r)$ rises from zero to a large value near $r = R$, where R

is loosely speaking the radius of the disk in which the electron system is confined. We choose this geometry, for which the electron system has a single edge, since we limit our attention here to the properties of an isolated quantum Hall edge and will not discuss the physics of interaction or scattering between edges [33]. In this geometry it is convenient to choose the symmetric gauge for which angular momentum is a good quantum number. For $V_{\rm conf}(r)$ the single-electron kinetic energy operator has the macroscopically degenerate Landau levels separated by $\hbar\omega_c$ and in each Landau level states with larger angular momentum are localized further from the origin. We recall from Eq.(4) that wavefunctions with angular momentum m are localized [6] near a circle with radius $R_m = \sqrt{2(m+1)}\ell$. (Note that for large m the separation between adjacent values of R_m is $\ell^2/R_m << \ell$.) In the strong magnetic field limit the confinement potential does not mix different Landau levels. Since there is only one state with each angular momentum in each Landau level the only effect of the confinement potential is to increase the energy of the symmetric gauge eigenstates when R_m becomes larger than $\sim R$. The typical situation is illustrated schematically in Fig. [5]. Here the $n = 0$ and $n = 1$ Landau levels are occupied in the bulk and the chemical potential μ lies in the gap $\Delta = \hbar\omega_c$ between the highest energy occupied Landau level ($E = 3\hbar\omega_c/2$) and the lowest energy unoccupied Landau level ($E = 5\hbar\omega_c/2$). In this section we are interested only in the ground state and the low energy excited states obtained by making one or more particle-hole excitations at the edge. We will discuss only the simplest situation where a single Landau level crosses the chemical potential at the edge of the system and the analogous *single branch* situations in the case of the fractional quantum Hall effect [34]. We will also neglect the spin degree of freedom of the electrons, which figured so prominently in the previous section.

An important property of the ground state of the non-interacting electron system in the case of interest, is that it remains an exact eigenstate of the system (but not necessarily the ground state!) when interactions are present. That is because the total angular momentum K for this state is

$$M_0 = \sum_{m=0}^{N-1} m = N(N-1)/2 \tag{13}$$

and all other states in the Hilbert space (truncated to the lowest Landau level) have larger angular momentum [35]. For large disks and total angular momentum near M_0 the excitation energy of a non-interacting electron state will be

$$\Delta E = \gamma M \tag{14}$$

where $M \equiv K - M_0$ is the excess angular momentum and γ is the energy separation between single-particle states with adjacent angular momenta and energies near the Fermi energy. γ is related to the electric field, $E_{\rm edge}$ from the confining potential at the edge of the disk:

$$\gamma = eE_{\rm edge}\frac{dR_m}{dm} = eE_{\rm edge}\ell^2/R \tag{15}$$

This expression for γ can be understood in a more appealing way. In a strong magnetic field charged particles execute rapid cyclotron orbits centered on a point that slowly drifts in the direction perpendicular to both the magnetic field and the local electric field. For an electron at the edge of the disk the velocity of this 'E cross B' drift is $v_{\rm edge} = cE_{\rm edge}/B$. The energy level separation can therefore be written in the form

$$\gamma = \hbar v_{\rm edge}/R_{\rm edge} = h/T \tag{16}$$

where T is the period of the slow drift motion of edge electrons around the disk, in agreement with expectations based on semiclassical quantization.

Since the excitation energy depends only on the angular momentum increase compared to the ground state it is useful to classify states by M. It is easy to count the number of distinct many-body states with a given value of M as illustrated in Fig. [6]. For $M = 1$ only one many-particle state is permitted by the Pauli exclusion principle; it is obtained by promoting the ground state electron with $m = N - 1$ to $m = N$. For $M = 2$, particle hole excitations are possible from $m = N - 1$ to $m = N + 1$ and from $m = N - 2$ to $m = N$. In general M many-particle states with excess angular momentum M can be created by making a single-particle hole excitation of the ground state. For $M \geq 4$ additional states can be created by making multiple particle-hole excitations. The first of these is a state with two particle-hole excitations that occurs at $M = 4$ and is illustrated in Fig. [6].

B. Many-Body Wavefunction Picture

We now discuss the edge excitation spectrum of interacting electrons using a language of many-particle wavefunctions. For the case of the integer quantum Hall effect we will essentially recover the picture of the excitation spectrum obtained previously for non-interacting electrons by counting occupation numbers. We could have used the Hartree-Fock approximation and occupation number counting to generalize these results to interacting electrons. However, the Hartree-Fock approximation is completely at sea when it comes to the fractional case. Discussions of the fractional edge using an independent electron language can be comforting but can also be misleading. Nevertheless, we will see that there is a one-to-one correspondence between the edge excitation spectrum for non-interacting electrons at integer filling factors and the fractional edge excitation spectrum.

Many-electron wavefunctions where all electrons are confined to the lowest Landau level must be sums of products of one-particle wavefunctions from the lowest Landau level. From Eq. (4) it follows that any N electron wavefunction has the form

$$\Psi[z] = P(z_1, \ldots, z_N) \prod_{\ell} \exp\left(-|z_\ell|^2/4\right), \tag{17}$$

where we have adopted ℓ as the unit of length and $P(z_1, \ldots, z_N)$ is a polynomial in the two-dimensional complex coordinates. This property [36] of the wavefunctions will be exploited in this section. The first important observation is that since $\Psi[z]$ is a wavefunction for many identical fermions it must change sign when any two particles are interchanged, and therefore must vanish as any two particles positions approach each other. Since $P(z_1, \ldots, z_N)$ is a polynomial in each complex coordinate it follows [37] that

$$P(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)\, Q[z] \tag{18}$$

where $Q[z]$ is any polynomial that is symmetric under particle interchange. It is important to note that the total angular momentum of all the particles (K) is just the degree of the polynomial $P[z]$, *i.e.* the sum of the powers to which the individual particle complex

coordinates are raised. Since the total angular momentum is a good quantum number the polynomial part of any many-electron eigenstate will be a homogeneous degree polynomial, *i.e.* all terms in the many-particle polynomial must have the same degree. Additionally, the total angular momenta corresponding to a polynomial that, as in Eq. (18), is the product of two polynomials is the sum of the angular momenta associated with those polynomials.

From the discussion of Section III it is clear that for non-interacting electrons in any monotonically increasing confinement potential, the lowest energy state will be the state with the minimum total angular momentum. In Eq. (18) that corresponds to choosing $Q[z]$ to have degree zero, *i.e.* to $Q[z] \propto 1$. It is easy to verify that the wavefunction when $Q[z]$ is a constant is in fact the Slater determinant formed by occupying the single-particle states with $m = 0, \ldots, N-1$. For interacting electrons this state will remain the ground state provided the confinement potential is strong enough to overcome the repulsive interactions between electrons that favor states with larger total angular momentum. When this is the ground state, low-energy excited states with excess angular momentum $M = K - N(N-1)/2$ are linear combinations of the states constructed by choosing all possible symmetric polynomials [38] of degree M for $Q[z]$.

We'll discuss the enumeration of these polynomials in a moment but pause now to explain how this analysis many be generalized to the case of the fractional quantum Hall effect. We limit our attention here [39] to the simplest fractional quantum Hall effects that occur at Landau level filling factors $\nu = 1/m$ for any odd integer m; in some senses the $m = 1$ case can be regarded as a special case of the fractional quantum Hall effect. The physics of the chemical potential jump that occurs at these filling factors was explained in the pioneering paper of Laughlin [14,6]. For $\nu < 1/3$, for example, it is possible to find states in the Hilbert space in which pairs of electrons are never found in a state with relative angular momentum equal to one. This is the two-body state in which two electrons are closest together. All many-particle states that avoid placing pairs in this state will have low energy. If this condition [40] is satisfied,

$$P(z_1, \ldots, z_N) \equiv \prod_{i<j} (z_i - z_j)^3 \, Q[z] \tag{19}$$

for any symmetric polynomial $Q[z]$. If the system has an abrupt edge the ground state will have $Q[z] \equiv 1$ just as in the non-interacting case. (Note that this approach to identifying the ground state and low-energy excitations is very much the same as that used in Section III to identify the ground state for ν near one when the electrons are not spin-polarized, except possibly spontaneously!) The edge excitations correspond to the same set of symmetric polynomials as in the $\nu = 1$ case. In the case of a model system with a short-range interaction and a parabolic confinement potential, it is easy to place [41] the argument we have sketched above on firm ground. It is known from numerical studies that the bulk chemical potential discontinuity survives when the model interaction is replaced by the realistic Coulomb interaction. However, special care is required in considering edge excitations in physically realistic systems with long range interactions and the reader is warned that the simple models discussed below may not always apply. These issues are discussed at greater length elsewhere [7].

The wavefunction

$$\Psi[z] \equiv Q[z] \prod_l \exp(-|z_l|^2/4) \tag{20}$$

is a wavefunction for N bosons in a strong magnetic field. Thus the enumeration of the edge excitations in terms of symmetric polynomials discussed above is equivalent to enumerating all many boson wavefunctions with a given value of the total angular momentum. The boson angular momentum

$$M = \sum_{m=0}^{\infty} m \, n_m \tag{21}$$

where n_m are the boson occupation numbers, is equivalent (for $\nu = 1/m$) to the excess angular momentum $M = K - mN(N-1)/2$ of the fermion wavefunctions. In the state with $M = 0$, $n_0 = N$, all other boson occupation numbers are zero, and $Q[z]$ is a constant. In the boson language the ground state is a Bose condensate. The lone state with $M = 1$ has $n_1 = 1$,$n_0 = N - 1$; the symmetric polynomial for this boson wavefunction is

$$Q[z] = z_1 + z_2 + \ldots z_N. \tag{22}$$

For the integer $\nu = 1$ case, it can be shown explicitly that the corresponding many-fermion state is the $M = 1$ state with a single particle-hole excitation at the edge, discussed in Section III. The set of excitations at general values of M can be described equally well in either fermion or boson languages. Some of the states that occur at small values of M are listed in Table I.

In the parabolic confinement case the total energy depends only on the excess total angular momentum: $\delta E = \gamma M$. The number of many-boson states with total angular momentum M, $g(M)$ can be calculated by considering a system of non-interacting bosons with single-particle energy γm so that $E = \sum_m (\gamma m) \cdot n_m = \gamma M$. The partition function is

$$Z = \sum_M g(M)\, e^{-\gamma M/k_B T} = \sum_M x^M\, g(M) \tag{23}$$

where $x = e^{-\gamma/k_B T}$. For $N \to \infty$, the $m = 0$ state acts like a reservoir with chemical potential $\mu = 0$ so that the partition function calculation can be done in the grand canonical ensemble. The degeneracies $g(M)$ can be read off the power series expansion of the partition function:

$$\begin{aligned} Z &= \prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots \\ &= (1+x+x^2+x^3+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\ldots \\ &= (1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\ldots. \end{aligned} \tag{24}$$

For [41] large M $g(M) \sim e^{\sqrt{\frac{2}{3}}\cdot\pi\cdot M^{1/2}}$. The function $g(M)$ is well known to number theorists from the theory of partitions [42] in which it is known as the partition function, not to be confused with the physics partition function above! For parabolic confinement potentials and short-ranged repulsive interactions, the degeneracy of the edge excitations at a given excess angular momentum is exact in both integer and fractional $\nu = 1/m$ cases. For general confinement potentials and general electron-electron interactions these degeneracies will be lifted. However, there is reason to expect that in the thermodynamic limit excitations with $M \ll N^{1/2}$ will be nearly degenerate. One way to see this is to use the chiral Luttinger liquid

picture of quantum Hall edges that we discuss in the following section. This approach will allow us to do more than enumerate excitations of the system and, in particular will enable us to discuss the density-of-states for tunneling into the edge of a quantum Hall system.

C. Chiral Luttinger Liquid Picture

The chiral Luttinger liquid picture [43] of quantum Hall systems is an adaptation of the Luttinger liquid theory of one-dimensional electron systems. We start this section with a brief outline of the portion of that theory that we require. Readers in search of greater depth should look elsewhere [44]. As in higher dimensions, low excitation energies states in a one-dimensional fermion system will involve only single-particle states near the Fermi wavevector. Since the differences in wavevector among the relevant states at a given Fermi edge are small, the excitations produced by rearranging them occur on length scales that are long compared to microscopic lengths. It is therefore reasonable to argue that the energy density in the system at any point in space should depend only on the local density of left-moving ($k < 0$) and right-moving ($k > 0$) electrons, $n_L(x)$ and $n_R(x)$:

$$E[n_L, n_R] = E_0 + \int dx \, \left[\frac{\alpha_{LL}}{2} \, \delta n_L^2(x) + \frac{\alpha_{RR}}{2} \, \delta n_R^2(x) + \alpha_{LR} \, \delta n_L(x) \, \delta n_R(x) \right]. \tag{25}$$

It is, perhaps, not completely obvious that the density provides a complete parameterization of the low-energy excitations, and indeed in the fractional Hall case there are situations where the analog of Eq. (25) is incorrect. [45] Here α_{LL}, α_{LR} and α_{RR} are determined by the second derivatives of the energy per unit length with respect to n_L and n_R for a uniform system and can be determined in principle by a microscopic calculation. $\delta n_L(x)$ and $\delta n_R(x)$ are differences of the density from the ground state density. Note that we have as a convenience chosen the chemical potential to be zero in dropping a term proportional to $\int dx(\delta n_L(x) + \delta n_R(x))$. We start by considering the case where $\alpha_{LR} = 0$ so that the left-moving electrons and right moving electrons are decoupled. Focus for this case on the energy of the right moving electrons. We Fourier expand the density and note that

$$\int dx\ \delta n_R^2(x) = \frac{1}{L}\sum_{q\neq 0} n_{-qR}.n_{qR} \tag{26}$$

so that the energy can be written in the form

$$E_R = E_0 + \frac{\alpha_{LL}}{2L}\sum_{q\neq 0} n_{-qR}n_{qR}. \tag{27}$$

The energy above can be used as an effective Hamiltonian for low-energy long-wavelength excitations. The simplification at the heart of the Luttinger liquid theory is the observation that when the Hilbert space is truncated to include only low-energy, long-wavelength excitations (in particular when the number of left-moving and right-moving electrons is fixed) Fourier components of the charge density do not commute. For example consider the second quantization expression for n_{qR} in terms of creation and annihilation operators with $k > 0$:

$$n_{qR} = \sum_{k>0} c^\dagger_{k+q} c_k. \tag{28}$$

An example of the dependence of the effect of products of these operators on the order in which they act is more instructive than the actual algebraic calculation of the commutators. Note for example that

$$n_{-qR}\,|\Psi_0\rangle = 0 \tag{29}$$

where $q > 0$ and $|\Psi_0\rangle$ is the state with all right-going electron states with $k < k_F$ occupied and all right-going states with $k > k_F$ empty. (The alert reader will have noticed that this state of 'right-going' electrons corresponds precisely to the 'maximum density droplet' states that occur in the quantum Hall effect.) n_{-qR} annihilates this state because there are no right-electron states with a smaller total momentum than $|\Psi_0\rangle$. On the other hand for $q = M2\pi/L$, $n_{qR}|\Psi_0\rangle$ yields a sum of M terms in which single-particle hole excitations have been formed in $|\Psi_0\rangle$. For example, if we represent occupied states by solid circles and unoccupied states by open circles, as in Fig. (6), for $M = 2$ we have

$$\begin{aligned} n_{qR}\,|\Psi_0\rangle = |\ldots \bullet\ \bullet\ \circ\ \bullet|\bullet\ \circ\ \circ\ldots\rangle \\ +|\ldots \bullet\ \bullet\ \bullet\ \circ|\circ\ \bullet\ \circ\ldots\rangle. \end{aligned} \tag{30}$$

Each of the M terms produced by $n_{qR}|\Psi_0\rangle$ is mapped back to $|\Psi_0\rangle$ by n_{-qR}. Therefore $n_{qR}n_{-qR}|\Psi_0\rangle = 0$ whereas $n_{-qR}n_{qR}|\Psi_0\rangle = M|\Psi_0\rangle$. The general form of the commutation relation is readily established by a little careful algebra [44]:

$$[n_{-q'R}, n_{qR}] = \frac{qL}{2\pi}\,\delta_{q,q'}. \tag{31}$$

This holds as long as we truncate the Hilbert space to states with a fixed number of right-going electrons and assume that states far from the Fermi edge are always occupied.

We can define creation and annihilation operators for density wave excitations of right-going electrons. For $q > 0$

$$a_q = \sqrt{\frac{2\pi}{qL}}\,n_{-qR} \tag{32}$$

$$a_q^\dagger = \sqrt{\frac{2\pi}{qL}}\,n_{qR} \tag{33}$$

With these definitions Eq. (31) yields

$$[a_{q'}, a_q^\dagger] = \delta_{q,q'} \tag{34}$$

so that the density waves satisfy bosonic commutation relations. Also note that

$$[\hat{M}, a_q] = -\frac{qL}{2\pi}a_q \tag{35}$$

$$[\hat{M}, a_q^\dagger] = \frac{qL}{2\pi}a_q^\dagger \tag{36}$$

where $\hat{M}$ is the total angular momentum operator. The contribution to the Hamiltonian from right-going electrons is therefore

$$H_R = \sum_{q>0} \hbar\, vq\, a_q^\dagger a_q \tag{37}$$

where

$$v = \frac{\alpha_{RR}}{2\pi\hbar} = \frac{1}{2\pi L\hbar}\,\frac{d^2E_0}{dn_R^2} = \frac{1}{2\pi\hbar}\,\frac{d\mu_R}{dn_R} \tag{38}$$

At low-energies the system is equivalent to a system of one-dimensional phonons traveling to the right with velocity v. In the limit of non-interacting electrons

$$v = \frac{\hbar k_F}{m^*} \equiv v_F \tag{39}$$

as expected.

Without interactions between left and right-moving electrons a Luttinger liquid is quite trivial. In particular the ground state ($|\Psi_0\rangle$) is a single-Slater determinant with a sharp Fermi edge. For one-dimensional electron gas systems the interesting physics [44] occurs only when left and right-moving electrons are allowed to interact. Most notably, arbitrarily weak interactions destroy the sharp Fermi edge that is the hallmark of Fermi liquids and that survives interactions in higher dimensions. In the case of quantum Hall edges, however, the above restriction to electrons moving in only one direction is not a temporary pedagogical device. The model with only right moving electrons discussed above can be taken over *mutatis mutandis* as a model of the edge excitations for an electron system with $\nu = 1$. The role played by the one-dimensional electron density is taken over by the integral of the two-dimensional electron density along a line perpendicular to the edge. In this way we arrive at the same bosonized picture of the ground state and low-lying excitations at the edge of a quantum Hall system as we reached previously by arguing in terms of many-particle wavefunctions. The single boson states which appeared there are replaced by the states of the chiral phonon system which has modes with only one sign of momentum and velocity.

For $\nu = 1$ the analysis applies whether or not the electrons interact. We now turn our attention to a discussion of the fractional case. Do all steps of the above discussion generalize? We can argue that if we are interested only in low-energy long-wavelength excitations, the energy can be expressed in the form

$$E = E_0 + \frac{\alpha}{2L} \sum_{q \neq 0} n_{-q} n_q. \tag{40}$$

As we comment later, this expression can fail at the edge of fractional quantum Hall systems although it is appropriate for $\nu = 1/m$. What about the commutator? There is an important difference in the line of argument in this case, since single-particle states far from the edge of the system are not certain to be occupied. Instead the average occupation number is

$\nu = 1/m$ and there are large quantum fluctuations in the local configuration of the system even in the interior. However, we know [39] from the discussion in terms of many-body wavefunctions in the previous section that the low-energy excitations at $\nu = 1/m$ *can* be described as the excitations of a boson system, exactly like those at $\nu = 1$, which suggests that something like Eq. (31) must still be satisfied when the Hilbert space is projected to low energies. If we replace the commutator by its expectation value in the ground state we obtain

$$[n_{-q'}, n_q] = \nu \cdot \frac{qL}{2\pi}\, \delta_{q,q'} \tag{41}$$

which differs from Eq. (31) only through the factor ν. It seems clear for the case of $\nu = 1/m$ this replacement can be justified on the grounds that the interior is essentially frozen (but in this case *not* simply by the Pauli exclusion principle) at excitation energies smaller than the gap for bulk excitations. What we need to show is that Eq. (41) applies as an operator identity in the entire low-energy portion of the Hilbert space. Below, however, we follow a different line of argument.

Appealing to the microscopic analysis in terms of many-body wavefunctions we know that the excitation spectrum for $\nu = 1/m$ is equivalent to that of a system of bosons. We conjecture that the commutator $[n_{-q'}, n_q] =\propto q\delta_{q,q'}$. To determine the constant of proportionality we will require that the rate of change of the equilibrium edge current with chemical potential be $e\nu/h$. From the edge state picture of the quantum Hall effect discussed in Section II, it is clear that this is equivalent to requiring the Hall conductivity to be quantized at $\nu e^2/h$. Since our theory will yield a set of phonon modes that travel with a common velocity v it is clear that the change in equilibrium edge current is related to the change in equilibrium density by

$$\delta I = ev\delta n. \tag{42}$$

When the chemical potential for the single edge system is shifted slightly from its reference value (which we chose to be zero) the grand potential is given by

$$E[n] = E_0 + \mu\delta n + \alpha\frac{(\delta n)^2}{2} \tag{43}$$

Minimizing with respect to δn we find that

$$\delta n = \frac{\delta\mu}{\alpha} \tag{44}$$

so that

$$\frac{\delta I}{\delta\mu} = \frac{ev}{\alpha} \tag{45}$$

In order for this to be consistent with the quantum Hall effect ($\delta I = (e\nu/h)\delta\mu$) our theory must yield a edge phonon velocity given by

$$v = \frac{\alpha}{h}\cdot\nu. \tag{46}$$

The extra factor of ν appearing in this equation compared to Eq. (38) requires the same factor of ν to appear in Eq. (41). We discuss below the qualitative changes in the physics [46,43] of fractional edge states which are implied by this outwardly innocent numerical factor.

It is worth remarking that the line of argument leading to this specific chiral Luttinger liquid theory of the fractional quantum Hall effect is not completely rigorous. In fact we know that this simplest possible theory with a single branch of chiral bosons does not apply for all filling factors [39,46,47], even though (nearly) all steps in the argument are superficially completely general. The reader is encouraged to think seriously about what could go wrong with our arguments. Certainly the possibility of adiabatically connecting all low-energy states with corresponding states of the non-interacting electron system, available for one-dimensional electron gases and for quantum Hall systems at integer filling factors but not at fractional filling factors, adds confidence when it is available. In our view, the microscopic many-particle wavefunction approach that establishes a one-to-one mapping between integer and fractional edge excitations (for $\nu = 1/m!$) is an important part of the theoretical underpinning of the Luttinger liquid model of fractional Hall edges. Once we

know that the edge excitations map to those of a chiral boson gas and that the fractional quantum Hall effect occurs, it appears that no freedom is left in the construction of a low-energy long-wavelength effective theory.

An important aspect of Luttinger liquid theory is the expression for electron field operators in terms of bosons [44]. This relationship is established by requiring the exact identity

$$[\rho(x), \hat{\psi}^\dagger(x')] = \delta(x - x')\ \hat{\psi}^\dagger(x') \tag{47}$$

to be reproduced by the effective low-energy theory. This equation simply requires the electron charge density to increase by the required amount when an electron is added to the system. The electron creation operator should also be consistent with Fermi statistics for the electrons:

$$\left\{\psi^\dagger(x), \psi^\dagger(x')\right\} = 0. \tag{48}$$

In order to satisfy Eq. (47), the field operator must be given by

$$\hat{\psi}^\dagger(x) = ce^{i\nu^{-1}\phi(x)} \tag{49}$$

where $d\phi(x)/dx = n(x)$ and c is a constant that cannot be determined by the theory. The factor of ν^{-1} in the argument of the exponential of Eq. (49) is required because of the factor of ν in the commutator of density Fourier components that in turn was required to make the theory consistent with the fractional quantum Hall effect. When the exponential is expanded the $k-th$ order terms generate states with total boson occupation number k and are multiplied in the fractional case by the factor ν^{-k}; multi-phonon terms are increased in relative importance. It is worth remarking [46] that the anticommutation relation between fermion creation operators in the effective theory is satisfied only when ν^{-1} is an odd integer. This provides an indication, independent of microscopic considerations, that the simplest single-branch chiral boson effective Hamiltonian can be correct only when $\nu = 1/m$ for odd m. Wen [43] has surveyed, using this criterion, the multi-branch generalizations of the simplest effective Hamiltonian theory which are possible at any given rational filling factor.

His conclusions are consistent with arguments [39] based on the microscopic theory of the fractional quantum Hall effect.

Eq. (49) has been carefully checked numerically [48] and appears to be correct. The ν^{-1} factor leads to predictions of qualitative changes in a number of properties of fractional edges. The quantity that is most directly altered is the tunneling density-of-states. Consider, for example, the state created when an electron, localized on a magnetic length scale, is added to the ground state at the edge of a $N-$ electron system with $\nu = 1/m$:

$$\begin{aligned}\hat{\psi}^\dagger(0)|\Psi_0\rangle &\sim \exp\left(-\sum_{n>0}\frac{a_n^\dagger}{\sqrt{n\nu}}\right)|\psi_0\rangle \\ &= 1 + \frac{\text{1 phonon term}}{\nu^{1/2}} + \frac{\text{2 phonon terms}}{\nu} + \ldots . \end{aligned} \tag{50}$$

The tunneling density states is given by a sum over the ground and excited states of the $N+1$ particle system:

$$A(\epsilon) = \sum_n \delta(E_n - E_0 - \epsilon)|\langle\Psi_n|\psi^\dagger(0)|\Psi_0\rangle|^2 \tag{51}$$

Because of the increased weighting of multiphonon states, which become more numerous at energies farther from the chemical potential, the spectral function is larger at larger $\epsilon - \mu$ in the fractional case. An explicit calculation [46,43] yields a spectral function that grows like $(\epsilon - \mu)^{\nu^{-1}-1}$. It is intuitively clear that the spectral function should be small at low-energies in the fractional case since the added electron will not share the very specific correlations common to all the low-energy states. It is amazing that by simply requiring the low-energy theory to be consistent with the fractional quantum Hall effect we get a very specific prediction for the way in which this qualitative notion is manifested in the tunneling density of states.

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[38] Michael Stone, Phys. Rev. B **42**, 8399 (1990); Michael Stone, Ann. Phys. (NY) **207**, 38 (1991); Michael Stone H.W. Wyld, and R.L. Schult, Phys. Rev. B **45**, 14156 (1992).

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and V. Meden, submitted to Am.J. Phys. (1995).

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FIGURES

FIG. 1. A large but finite two-dimensional electron gas. In panel (a) the chemical potential lies in a gap and the only low-energy excitations are localized at the edge of the system. In panel (b) the chemical potential lies in a mobility gap so that there are low-energy excitations in the bulk but they are localized away from the edge. In panel (c) a net current is carried from source to drain by having local equilibria at different chemical potentials on upper and lower edges.

FIG. 2. Illustration of a Skyrmion spin texture. At the center of the Skyrmion $\hat{m}$ points in the down ($-\hat{z}$) direction. Far from the center of a Skyrmion $\hat{m}$ points in the up ($\hat{z}$) direction. Along a ray at angle θ in a circular coordinate system defined with respect to the Skyrmion center, $\hat{m}$ rotates about an axis in the $(\sin(\theta) - \cos(\theta)$ direction from $-\hat{z}$ to $\hat{z}$. At fixed r the $\hat{x} - \hat{y}$ projection of $\hat{m}$ has fixed magnitude and rotates by $\pm 2\pi$ when the angular coordinate winds by $\pm 2\pi$. At $r = \lambda$, $\hat{m}$ lies entirely in the $\hat{x} - \hat{y}$ plane.

FIG. 3. Knight shift measurements by Barrett *et al.* of the spin polarization of a two-dimensional electron gas near filling factor $\nu = 1$. Here $S = A = K + 1/2$ so that the experiment is consistent with $K = 3$ for this sample. The dashed line in this figure shows the dependence of spin-polarization on filling factor expected for non-interacting electrons and, in the Hartree-Fock approximation, also for interacting electrons. The spin-polarization is assumed to be proportional to the Knight shift of the ^{71}Ga nuclear resonance and to be complete at $\nu = 1$. (After Ref. [28]

FIG. 4. Dependence of spin-polarization P on filling factor for Skyrme lattice states. Here g is the ratio of the Zeeman energy to the characteristic interaction energy e^2/ℓ and the values chosen are typical of experimental systems. The open and closed circles are experimental results of Barrett *et al.*. The legends indicate the nature of the Skyrme lattice state: the SLA state is a square lattice state with opposing Skyrmion orientations; the TLF state is a triangular lattice state with aligned Skyrmion orientations. (After Brey *et al.* in Ref. [32])

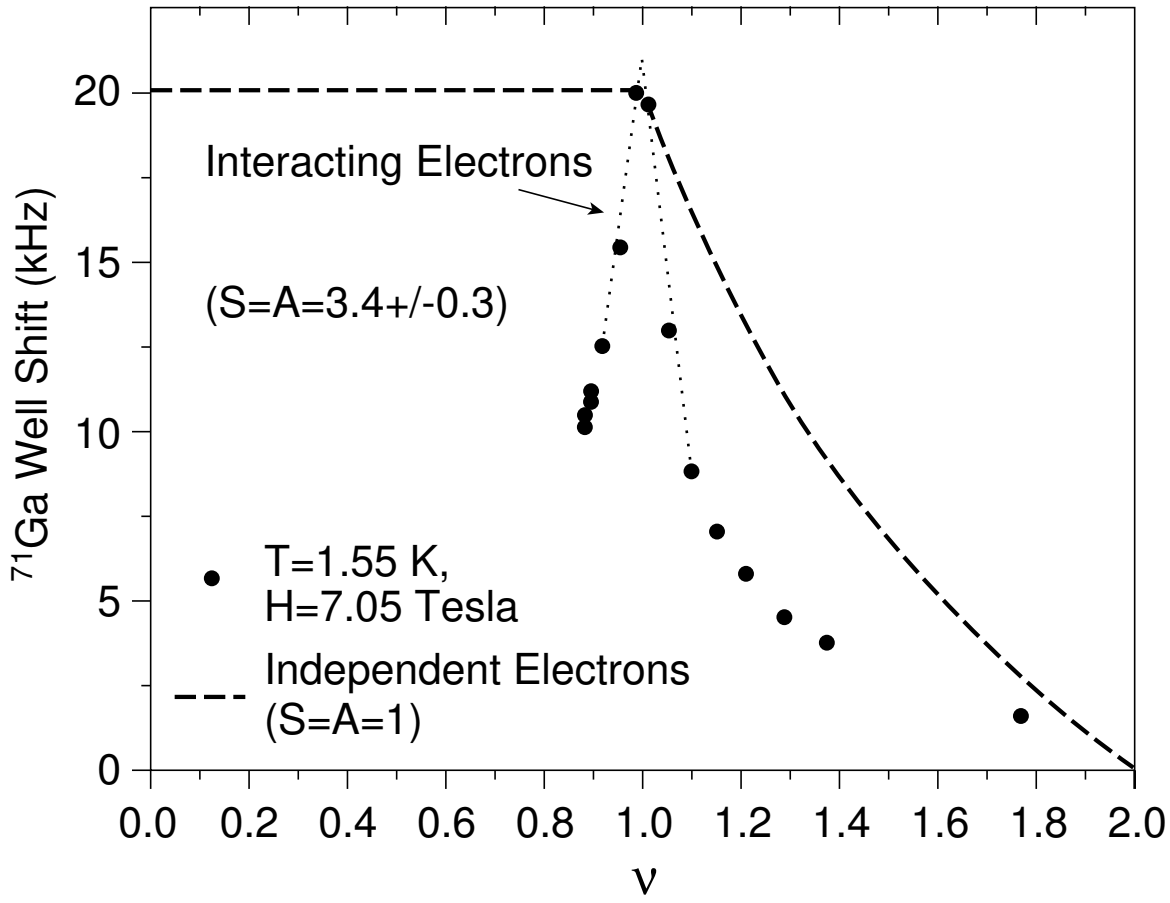
FIG. 5. Schematic spectrum for non-interacting electrons confined to a circular disk in a strong magnetic field. In the limit of large disks the dependence of the energy on m can usually be considered to be continuous. The situation depicted has Landau level filling factor $\nu = 2$ in the bulk of the system. The low-energy excitations are particle-hole excitations at the edge of the system.

FIG. 6. Non-interacting many electron eigenstates for small excess angular momentum M specified by occupation numbers for the single-particle states with energies near the chemical potential μ. The vertical bars separate single-particle states with $\epsilon_m < \mu$ from those with $\epsilon_m > \mu$. A solid circle indicates that $n_m = 1$ in both the ground state and in the particular excited state; a shaded circle indicates that $n_m = 1$ in the particular excited state but not in the ground state; an empty circle indicates that $n_m = 0$.

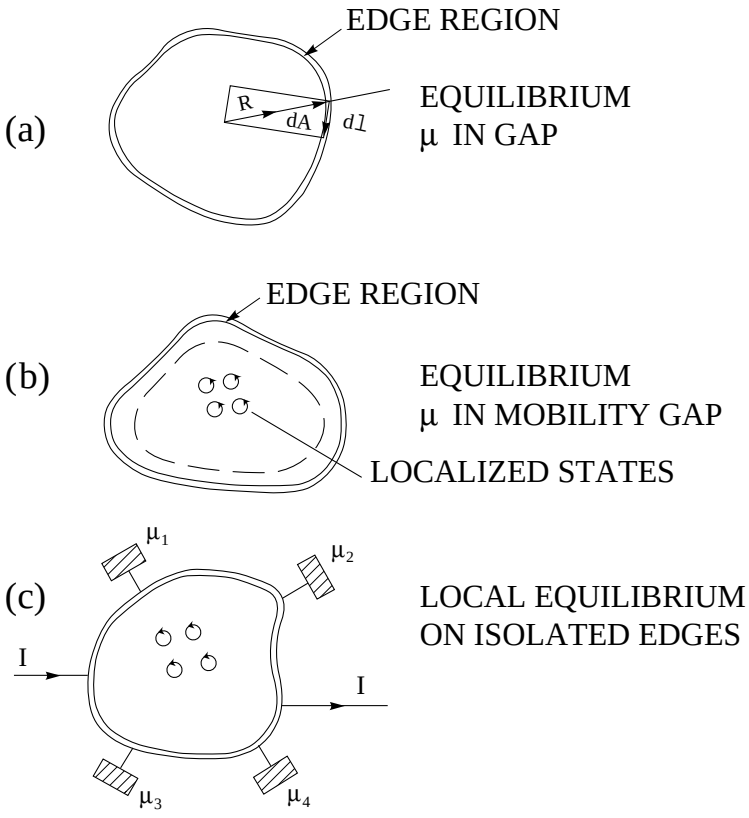
TABLES

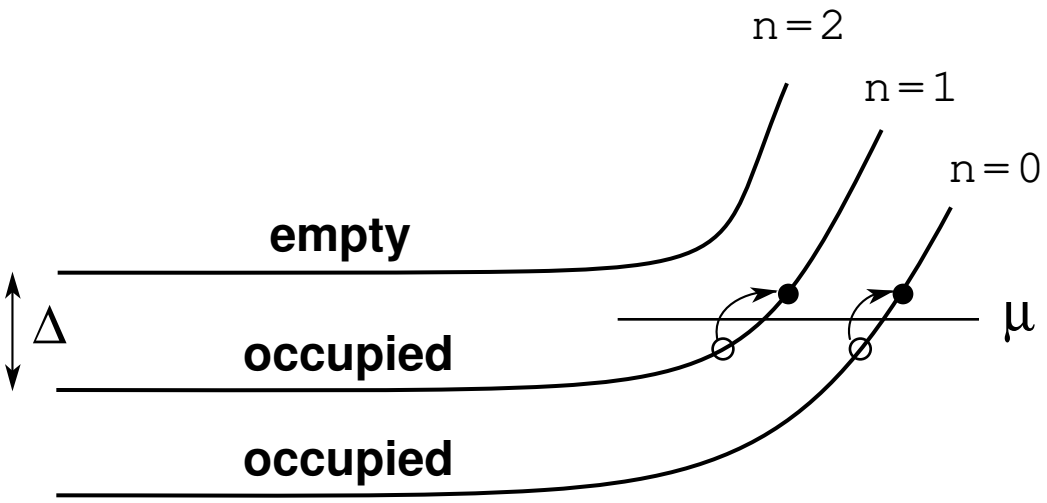
TABLE I. Quantum occupation numbers in boson and fermion descriptions for edge excitations with small excess angular momentum M. g_M is the number of states with excess angular momentum M. The fermion occupation numbers are relative to the maximum density droplet state. Only non-zero values are listed for both fermion and boson descriptions. $L = N - 1$ is the highest angular momentum that is occupied in the maximum density droplet state.

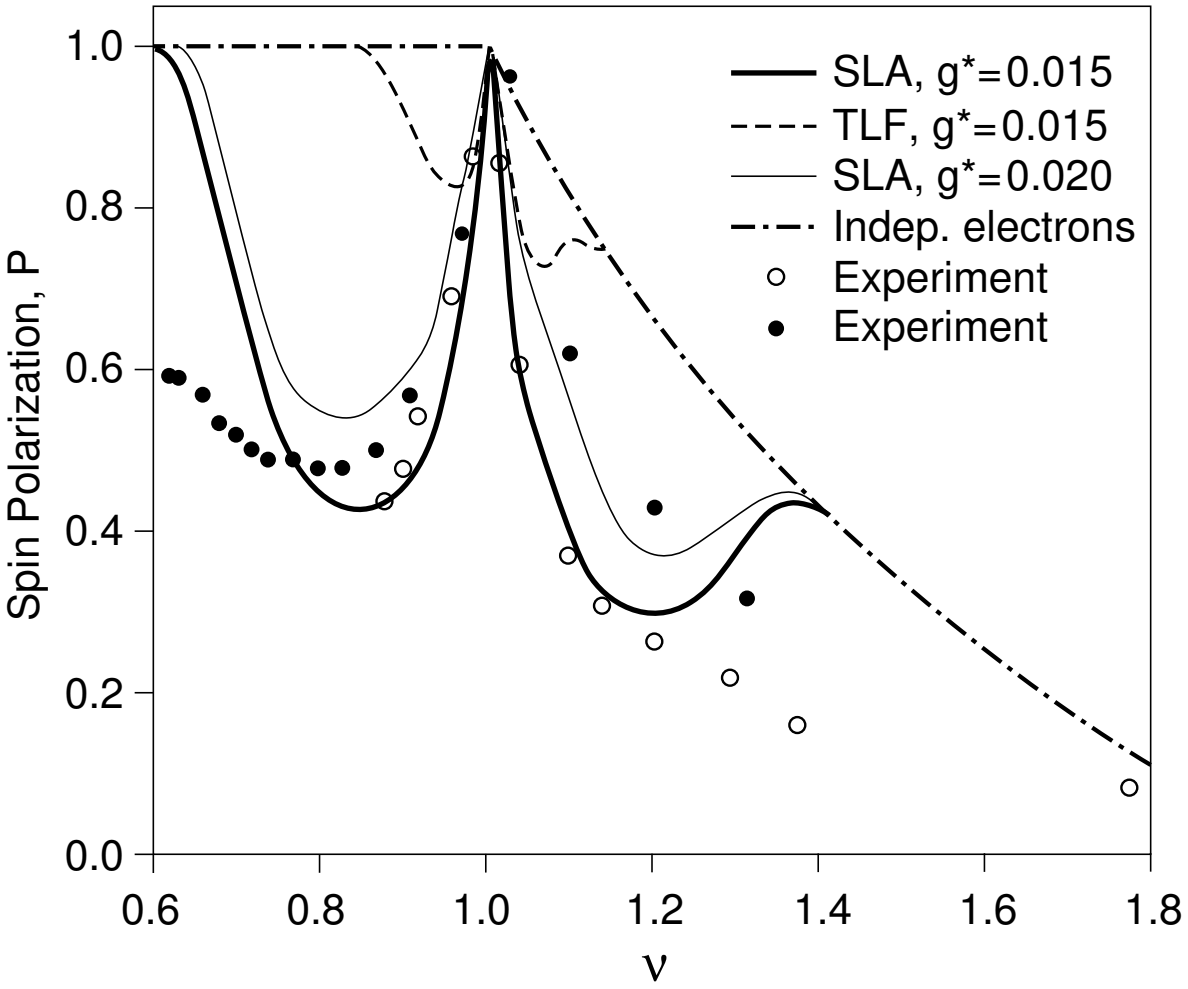
M	g_M	Fermion Description	Boson Description
1	1	$n_{L+1} = 1, n_L = -1$	$n_1 = 1$
2	2	$n_{L+2} = 1, n_L = -1; n_{L+1} = 1, n_{L-1} = -1$	$n_2 = 1; n_1 = 2$
3	3	$n_{L+3} = 1, n_L = -1; n_{L+2} = 1, n_{L-1} = -1$	$n_3 = 1; n_2 = 1, n_1 = 1;$
		$n_{L+1} = 1 n_{L-2} = -1$	$n_1 = 3$
4	5	$n_{L+4} = 1, n_L = -1; n_{L+3} = 1, n_{L-1} = -1$	$n_4 = 1; n_3 = 1, n_1 = 1; n_2 = 2$
		$n_{L+2} = 1, n_{L-2} = -1; n_{L+1} = 1, n_{L-3} = -1$	$n_2 = 1, n_1 = 2; n_1 = 4$
		$n_{L+2} = 1, n_{L+1} = 1, n_L = -1, n_{L-1} = -1$	


20
15
10
5
0
^{71}Ga Well Shift (kHz)
Interacting Electrons
(S=A=3.4+/-0.3)
T=1.55 K,
H=7.05 Tesla
Independent Electrons
(S=A=1)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
ν


• M = 1
. . . ● ● ● ● ○ | ◍ ○ ○ ○ . . .
$g_{M=1} = 1$
• M = 2
. . . ● ● ● ○ ● | ◍ ○ ○ ○ . . .
. . . ● ● ● ● ○ | ○ ◍ ○ ○ . . .
$g_{M=2} = 2$
• M = 4
. . . ● ● ● ○ ○ | ◍ ◍ ○ ○ . . .
$g_{M=4} = 5$


(a)
EDGE REGION
R
dA
dl
EQUILIBRIUM
μ IN GAP
(b)
EDGE REGION
EQUILIBRIUM
μ IN MOBILITY GAP
LOCALIZED STATES
(c)
μ1
μ2
I
I
μ3
μ4
LOCAL EQUILIBRIUM
ON ISOLATED EDGES


n = 2
n = 1
n = 0
empty
Δ
occupied
occupied
μ


Spin Polarization, P
1.0
0.8
0.6
0.4
0.2
0.0
0.6
0.8
1.0
1.2
1.4
1.6
1.8
ν
SLA, g*=0.015
TLF, g*=0.015
SLA, g*=0.020
Indep. electrons
Experiment
Experiment