

A Simple Formula for the Persistent Current in
Disordered 1D Rings: Parity and Interaction Effects

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The general expression for the persistent current of 1D noninteracting electrons in a disorder potential with smooth scattering data is derived for zero temperature. On the basis of this expression the parity effects are discussed. It is shown that the electron-electron interaction may give rise to an unusual size-dependence of the current amplitude due to renormalized backscattering from the disorder potential.

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The persistent current in 1D mesoscopic rings, being a manifestation of the Aharonov-Bohm effect for many-body systems, addresses physical questions of principle. Although such an effect was predicted several decades ago [1], in normal state systems it was only recently discovered experimentally. The experiments were performed on ensembles of many mesoscopic rings [2] as well as on single (or several) rings [3]. Generally, the theory predicts a magnitude of the current, which is much less than one experimentally observed [4], although there are some cases for which the theory and experiment seem to agree [5]. To the best of our knowledge there are no such studies of the interplay between the disorder and electron-electron interaction in 1D which would take into account nonperturbative effects in the persistent current (on the other hand, several numerical investigations have been carried out [5]).

The aim of this paper is to derive a simple formula for the persistent current of noninteracting electrons in a single-channel mesoscopic ring with a disorder potential and to study the interplay between the disorder and electron-electron interaction in 1D.

For N spinless electrons on the ring of the length L under Aharonov-Bohm conditions the persistent current is given by the formula:

$$\begin{aligned}
 j(\varphi) &= j_0(\varphi) + j^{par}(\varphi) + O(1/L^2); \\
 j_0(\varphi) &= \frac{v_F}{\pi L} \frac{\sqrt{T_F} \sin \varphi}{\sqrt{1 - T_F \cos^2 \varphi}} F(p_F, \varphi); \quad F(p, \varphi) = \cos^{-1} \left(\sqrt{T(p)} \cos \varphi \right), \quad (1) \\
 j^{par}(\varphi) &= -\frac{v_F}{L} \frac{\sqrt{T_F} \sin \varphi}{\sqrt{1 - T_F \cos^2 \varphi}}, \quad (N = \text{even}); \quad j^{par}(\varphi) = 0, \quad (N = \text{odd}),
 \end{aligned}$$

where $\varphi = 2\pi\Phi/\Phi_0$ (Φ is the magnetic flux passing through the ring) and $T_F \equiv T(p_F)$

is the transmission coefficient of the ring at the Fermi energy. The above formula explicitly shows that the persistent current is a purely Fermi surface effect. Moreover, it is parametrized by a single number - T_F (i.e. the only characteristic of the potential, which determines the flux dependence of the current, is T_F) [6].

Derivation. The single electron Schrödinger equation

$$\{\varepsilon_0(-i\partial_x) + V(x) - \varepsilon\}\psi = 0, \quad (2)$$

where $V(x)$ is the disorder potential and $\varepsilon_0(p)$ is the dispersion law, should be solved under the twisted boundary condition: $\psi(x+L) = e^{i\varphi}\psi(x)$. It was recognized already in the pioneer papers [1] that in this Bloch functions problem the whole ring plays the role of the elementary cell and φ - the role of the quasimomentum. The ground state energy as a function of the flux is given by the sum $E_0(\varphi) = \sum \varepsilon_\lambda(\varphi)$ over N lowest values of the band spectrum at fixed φ .

Suppose, for clarity, that the potential $V(x)$ is localized within the region of a radius $a < L$ (actually, the relation between a and L will be shown to be unimportant). The wave-function $\psi(x)$ can be then written in the form $\psi(x) = Ae^{ipx} + Be^{-ipx}$ from the left of the potential and in the form $\psi(x) = Ce^{ipx} + De^{-ipx}$ from the right of it. The coefficients C and D can be expressed through A and B by making use of the scattering data of $V(x)$ (transfer matrix). The twisted boundary condition gives then (after elementary algebra) the equations for the spectrum:

$$pL = \Phi_+(p, \varphi) \quad (n = 0); \quad pL = 2\pi n + \Phi_\pm(p, \varphi) \quad (n = 1, 2, \dots); \quad (3)$$

where $\Phi_\pm(p, \varphi) = \delta(p) \pm F(p, \varphi)$ and $\delta(p)$ is the forward scattering phase defined by

the scattering solution: if $A = 1$ and $D = 0$, then $C = \sqrt{T(p)}e^{i\delta(p)}$.

To make progress with the Eq.(3), the idea is to expand the solution in $1/L$. This is simplest when $a \ll L$ (e.g. $V(x)$ represents a single scatterer) and $\varepsilon_0(p) = p^2/2m$.

Then one can write:

$$p_n = k_n + \frac{1}{L}\Phi_{\pm}(k_n) + \frac{1}{L^2}\Phi_{\pm}(k_n)\frac{\partial\Phi_{\pm}(k_n)}{\partial k} + O\left(\frac{1}{L^3}\right); \quad k_n = 2\pi n/L \quad (4)$$

For the ground state energy we therefore have:

$$E_0 = \frac{1}{m} \sum_n \left\{ k^2 + \frac{2}{L}k\delta(k) + \frac{1}{2L^2}\frac{\partial}{\partial k} \left[k \sum_{\pm} \Phi_{\pm}^2(\varphi, k) \right] + O\left(\frac{1}{L^3}\right) \right\} \Big|_{k=k_n} \quad (5)$$

Here the first term is the ground state energy in the absence of the potential and flux (it is proportional to the volume: $\sim L$). The second term (~ 1) is the energy difference due the scattering potential (in agreement with Fumi's theorem [7]), which is flux independent. Thus, the effect of the flux is of order $1/L$ and is given by:

$$\begin{aligned} \Delta E_0(\varphi) &= E_0(\varphi) - E_0(0) = \sum_n \left\{ \frac{1}{2L^2}\frac{\partial}{\partial k} \left[k \sum_{\pm} \Phi_{\pm}^2(\varphi, k) - \Phi_{\pm}^2(0, k) \right] + O\left(\frac{1}{L^3}\right) \right\} \Big|_{k=k_n} = \\ &= \frac{v_F}{2\pi L} \left\{ F^2(p, \varphi) - F^2(p, 0) \right\} + O\left(\frac{1}{L^2}\right). \end{aligned} \quad (6)$$

For $N = \text{even}$ one additional particle on top of the spectrum contributes to Eq.(5) the term:

$$\Delta E_0^{par}(\varphi) = -\frac{v_F}{L} \{ F(p, \varphi) - F(p, 0) \} \quad (7)$$

In fact, each particle contributes to the flux dependence of the energy a term $\sim 1/L$, but the contributions of the particles with quantum numbers $(n, +)$ and $(n, -)$, which would correspond to the momenta $+p$ and $-p$ for $V = 0$, almost cancel each other,

and the entire contribution of the Fermi sea is again of the order of $1/L$ and converges actually just at the Fermi surface (Eq.(6)). For even N , the particle on the top, i.e. in the state $(N/2, -)$, does not have a partner in the state $(N/2, +)$, so the contribution of this single particle is $1/L$; that gives rise to the parity effect (Eq.(7)).

The formula Eq.(1) follows from Eqs.(6,7) by a standard definition of the current $j(\varphi) = -\partial E_0(\varphi)/\partial\varphi$. The diffusion coefficient $D = -\frac{1}{2}(\partial^2 E_0(\varphi)/\partial^2\varphi)|_{\varphi=0}$ is equal to D_0 for odd N and to $D_0 + D^{par}$ for even N , where

$$D_0 = \frac{v_F}{2\pi L} \sqrt{\frac{T_F}{1-T_F}} \tan^{-1} \sqrt{\frac{1-T_F}{T_F}}; \quad D^{par} = -\frac{v_F}{2L} \sqrt{\frac{T_F}{1-T_F}}. \quad (8)$$

Thus, for the case of spinless fermions, the ground state is always diamagnetic for odd N and paramagnetic for even N . In the absence of spin-orbit coupling the spin effect can be straightforwardly taken into account. The effect of the Fermi sea, i.e. the quantities $j_0(\varphi)$, $\Delta E_0(\varphi)$ and D_0 , should be just multiplied by the factor 2. The parity effect, however, changes a little bit: the diamagnetic ground state ($D > 0$) is observed only if $N = 4k + 2$. So, the definite conclusion concerning a dia- or paramagnetic nature of the ground state can be indeed drawn from the parity arguments only, even in the presence of a disorder potential [8].

The transfer matrix method has already been used in the theory of persistent current [9]. However, it is a combination of the transfer matrix method and the $1/L$ - expansion, which leads to the new results presented above. For a single point scatterer $V(x) = \varepsilon\delta(x)$ the transmission coefficient is $T_F = p_F^2/(p_F^2 + m^2\varepsilon^2)$. Expanding our formula Eq.(1) either in the parameter $m\varepsilon/p_F$ (weak potential) or in the parameter $p_F/m\varepsilon$

(strong potential) we reproduce the results for these limiting cases, which were obtained previously [9]. The condition $a \ll L$ assumed above for simplicity is unimportant. Actually, in the case of $a \sim L$ the derivation is essentially the same, except of the modification due to the fact that the forward scattering phase $\delta(p)$ in Eq.(3) is now $\sim L$. So, one should redefine the momentum according to $pL = 2\pi n + \delta(p)$ first and only then expand the effect of the flux in $1/L$: the results Eq.(1,6-8) will not change at all. Simple algebra also shows that the results are valid for arbitrary dispersion law $\varepsilon_0(p)$ (T_F is then understood to be defined accordingly to the dispersion law). The only important condition for the potential $V(x)$, which was in fact assumed in the above derivation, is the requirement for the transmission coefficient $T_F(p)$ to be a smooth function of the momentum on the scale $1/L$ (otherwise $dT_F(p)/dp$ becomes $\sim L$ and the $1/L$ - expansion obviously breaks down). Thus Eq.(1) works for the case of arbitrary single scatterer (or several ones) and, even under strong localization conditions (mean free path $l \ll L$), gives a correct order of magnitude for the current [9] and predicts also a $\sim \sin \varphi$ shape of its flux dependence, which was observed in numerical simulations for strong disorder [5].

In what follows we consider the effects due to electron-electron interactions (we consider the case of a single scatterer; note however, that in the long wave-length limit $q \rightarrow 0$ corresponding to the Luttinger Liquid fixed point we need only the disorder potential to be confined within the distance $1/q$). We make use of Haldane's [10] concept of topological excitations in the LL ground state in the boson representation. Due to the linear dispersion law near the Fermi points bosonic excitations do not

contribute to the current, and j is entirely defined by the topological number J which describes the difference between the number of right- and left-moving electrons

$$j = -\frac{\partial E_0}{\partial \varphi} = \frac{v_J}{L} \left(J - \frac{\varphi}{\pi} \right), \quad (9)$$

where $E_0(\varphi)$ is the ground state energy of the Hamiltonian [10]

$$H = \frac{\pi v_J}{2L} \left(J - \frac{\varphi}{\pi} \right)^2. \quad (10)$$

In this formulation the problem is a classical one: to find the minimum of the potential energy Eq.(10) with an integer J satisfying the parity condition $(-1)^J = (-1)^{N+1}$.

In the boson representation the backscattering Hamiltonian has a form

$$H_{bsc} = V_b \sum_{J=-\infty}^{+\infty} \left(a_{J+2}^+ a_J e^{i\hat{\theta}} + a_{J-2}^+ a_J e^{-i\hat{\theta}} \right), \quad \hat{\theta} = \sqrt{g} \sum_q \sqrt{\frac{2\pi}{L|q|}} (b_q^+ + b_q), \quad (11)$$

where we introduce a_J^+ - the creation operator of the quantum number J ; b_q^+ is the boson creation operator, and g is the dimensionless electron-electron coupling constant ($g = 1$ for the noninteracting system; $g > 1$ for the attractive interaction, and $g < 1$ for the case of repulsion).

Every backscattering event changes the topological number J by ± 2 with a simultaneous excitation of the bosonic environment. Now J is no longer a conserved quantity but has a tendency toward delocalization into a finite-size band. As we show below, the delocalization of J over the scale $\langle J^2 \rangle = R \gg 1$ leads to an exponential suppression of the current $\ln j \sim -R$. One might consider the effect of delocalization as equivalent to heating the perfect system up to the temperature $T \sim R 2\pi v_J / L$. The crucial point is that the parabolic potential Eq.(10) is very weak (of order of $1/L$) as compared with

the bare "hopping" rate V_b . If the dissipation effects are neglected ($g \rightarrow 0$) then in the ground state one finds the result $R \sim \sqrt{L}$. However this is never the case because bosonic environment plays the dominant role in the formation of the ground state.

The Hamiltonian Eq.(11) is well known in the theory of quantum coherence for so-called Ohmic dissipative environments (see, e.g. [12]). The effective bandwidth (or kinetic energy) corresponding to Eq.(11) is defined from the self-consistent equation

$$\Delta \approx V_b \left(\frac{\Delta, \omega_{min}}{E_F} \right)^g, \quad (12)$$

where E_F is the Fermi energy, and $\omega_{min} \approx 2\pi v_s/L$ is the minimal boson energy. In the simplest case $g > 1$ we immediately find that the kinetic energy is proportional to $\Delta \sim 1/L^g$ and can be neglected as compared with the potential term (10). As expected, we recover the known result that backscattering is irrelevant for the case of attractive interaction between the electrons, in this picture, due to localization of J .

For $g = 1$ it follows from (12) that Δ is proportional to $1/L$. Again, J is localized (in a sense that fluctuations of J are of order 1), but this time the solution depends on the details of the potential, electron energy spectrum etc. in agreement with the above discussion for the noninteracting electrons.

In the most intriguing case $g < 1$ the coherence is restored between the neighbouring values of J at frequencies below

$$\Delta \approx E_F \left(\frac{V_b}{E_F} \right)^{1/(1-g)}, \quad (13)$$

with delocalization of J in the ground state. Let us start with the extreme case of $g = 0$. In the "momentum" representation the Hamiltonian (10)-(11) can be conveniently

rewritten in the form (one might identify the momentum p with the canonical conjugate phase field $\bar{\theta}_J$; $[J, \bar{\theta}_J] = i$, see Ref. [10]):

$$H = -\frac{2\pi v_J}{L} \frac{\partial^2}{\partial p^2} + 2\Delta \cos p, \quad (14)$$

with twisted periodic condition $\psi(p + 2\pi) = e^{i\varphi}\psi(p)$ (we assumed $J = \text{even}$ here; the case of $J = \text{odd}$ will correspond to $\varphi \rightarrow \varphi + \pi$). Since the effective mass in the momentum representation is proportional to $1/L$ we can consider this Hamiltonian as a standard tight-binding model with the energy of the lowest state being defined as $E_0(\varphi) = \text{const} + D(1 - \cos \varphi)$, and the amplitude D being calculated in the semiclassical approach

$$D \approx \sqrt{\frac{8v_J \Delta}{\pi L}} \exp\left(-8\sqrt{\frac{L\Delta}{2\pi v_J}}\right). \quad (15)$$

Thus, for the case $g = 0$, the persistent current, $j \approx D(-1)^N \sin \varphi$, would be exponentially suppressed.

With $g > 0$, however, the concept of the effective mass for J completely fails even in the ground state. The dynamics of J is overdamped as follows from the constant mobility in the zero-temperature limit [13], $\mu = 1/(2\pi g)$. In the presence of the parabolic potential Eq.(10) the delocalization of J in the ground state depends on L only logarithmically [14], $\langle J^2 \rangle = \frac{\mu}{\pi} \ln(L\Delta/(2\pi v_J \mu^2))$, and one might expect a power dependence $J(L)$. It is easy to show [15], that in the momentum representation the problem can be reduced to the study of the effective action $\int Dp \exp(-S_{eff})$

$$S_{eff} = \int_{-\beta/2}^{\beta/2} d\tau \ 2\Delta \cos p(\tau) + \beta \sum_n \frac{\omega_n^2 |p_n|^2}{2(4\pi v_J/L + 2\pi g |\omega_n|)}; \quad (\beta \rightarrow \infty), \quad (16)$$

where $\omega_n = 2\pi n/\beta$, and p_n is the Fourier-transform of $p(\tau)$. Except very low frequencies $\omega_n \leq 2v_J/gL$ the effective action is governed by the dissipative term $\frac{\beta}{4\pi g} \sum |\omega| |p_\omega|^2$. The quasiclassical solution for the overdamped motion in the cos-barrier was studied in Ref. [16], so we can readily write down the expression for the current amplitude D in the leading logarithmic approximation

$$D \sim g\Delta \exp\left(-\frac{1}{g} \ln\left[\frac{L\Delta}{2\pi v_J}\right]\right). \quad (17)$$

Thus our basic result for the persistent current in the case of repulsion between the electrons has a form

$$j \sim g\Delta \left(\frac{2\pi v_J}{L\Delta}\right)^{1/g} (-1)^N \sin \varphi. \quad (18)$$

The suppression of the current to higher order in $1/L$ is closely related to the result of Ref. [11] that in the limit $E \rightarrow E_F$ backscattering renormalizes into a perfect reflection from the potential. Effectively the transmission coefficient near the Fermi points has a power law dependence [11]

$$T(E) \sim T_0 \left(\frac{E - E_F}{\Delta}\right)^{2(1/g-1)}, \quad (19)$$

leading to $T_F \sim T_0 L^{-2(1/g-1)}$. If we substitute this result into the formula Eq.(??), then in the limit $T_F \ll 1$ we easily obtain the same scaling behavior Eq.(18) [17].

To summarize, we derive for the first time an explicit analytic solution for the persistent current of noninteracting electrons in disordered 1D ring under quite general assumptions about the disorder potential. We find that the transmission coefficient of the ring at the Fermi energy is the only relevant parameter in the problem. The case of

spinless electrons is solved including electron-electron interactions, and a nontrivial size dependence of the persistent current with interaction-dependent exponent is obtained (provided the parameter Δ , Eq. (13), is large enough compared with $v_j 2\pi/L$). It seems interesting to generalize the above approach to the case of spin-orbit coupling and to the (more realistic) case of multi-channel rings for which the existence of a formula expressing the persistent current in terms of the transmittance matrix at the Fermi energy is still expected (although the flux dependence of the current will be more complicated). These generalizations are now in progress [15].

We are thankful to P.C.E.Stamp, A.G.Aronov, H.Cappelmann, A.Ioselevich, A.I.Larkin and M.Fabrizio for valuable discussions. One of us (N.V.P.) was supported by NSERC.

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